Variational and Topological Methods: Theory, Applications, Numerical Simulations, and Open Problems (2012). *Electronic Journal of Differential Equations*, Conference 21 (2014), pp. 149–172. ISSN: 1072-6691. http://ejde.math.txstate.edu, http://ejde.math.unt.edu ftp ejde.math.txstate.edu

A SURVEY OF RESULTS AND OPEN PROBLEMS FOR THE HYDRODYNAMIC FLOW OF NEMATIC LIQUID CRYSTALS

JAY LAWRENCE HINEMAN

Abstract. The mathematical study of the hydrodynamic flow of nematic liquid crystals is a mix of material science, Navier-Stokes theory, and the study of harmonic maps. In this article, the basic model for the hydrodynamic flow of nematic liquid crystals is presented – that is, the Ericksen-Leslie equations. There are a number of simplifications of the Ericksen-Leslie equations exist in the literature. We give a short survey of the known mathematical results for the Ericksen-Leslie equations and its simplifications. Along the way, open problems in analysis and computation for the Ericksen-Leslie system are discussed.

1. Introduction

Liquid crystals and their mathematical description have a long and rich history dating back over 100 years. Liquid crystals can be thought of as materials that exhibit an intermediate phase between liquid and solid. That is, while liquid crystals may flow like liquids, they exhibit additional structural properties.

Many chemical compounds have liquid crystal phases. For example concentrated solutions of rigid polymers in suitable solvents, DNA, and certain viruses all exhibit liquid crystal phases. Since there are many possible microscopic structures, there are accordingly many liquid crystal phases, for example, nematic, smectic, and cholesteric.

The simplest example of a liquid crystal phase occurring in nature is the nematic phase for a single chemical species. Physically, in the single-species case, liquid crystals that exhibit a nematic phase, or nematic liquid crystals for short, are liquids that are uniformly composed of rod-like molecules whose structure induces a preferred average directional order. A historical example is the compound MBBA, *N-(p-methoxybenzylidene)-p-butylaniline*, which is in the nematic phase for approximately the range temperatures $20°C$ to $47°C$ and whose length is on the scale of Angstroms.

For the modeling of a single species of a nematic at a fixed temperature one can consider a continuum theory which disregards the individual molecular structure. Such a continuum assumption is valid since the distance over which directional

2000 *Mathematics Subject Classification.* 76A15, 58E20.
Key words and phrases. Liquid crystals; Ericksen-Leslie; harmonic maps.

Published February 10, 2014.

order occurs is much larger than the molecular dimensions (that is proportional to μm versus proportional to Angstroms).

For a more in depth and complete introduction to liquid crystals, their history, and their mathematical study one should consult the texts of Virga [54], DeGennes and Prost [17], and or Stewart [51].

This survey we will focus the mathematical theory for the hydrodynamic flow of nematic liquid crystals. Specifically we will examine the Ericksen-Leslie equations for the hydrodynamic flow of nematic liquid crystals and simplifications. We begin in the next section with an outline of the derivation of the Ericksen-Leslie equations from balance laws. After this we discuss results for modifications and simplifications of the equations. The Ericksen-Leslie equations have three major facts that make their analysis difficult.

(1) They are a strongly coupled system of nonlinear equations – they contain a Navier-Stokes like system and a nonlinear system for the transported heat flow of harmonic maps.
(2) They contain physically relevant terms beyond those in the standard Navier-Stokes equations and transported heat flow of harmonic maps.
(3) There is a natural, point-wise nonlinear constraint on the system.

The first fact cannot be avoided and it is noted that in general that mathematical results about the well-posedness of the Ericksen-Leslie equations cannot surpass those for Navier-Stokes. Simplification of the Ericksen-Leslie equations have been employed to deal with the second and third facts. Specifically, the difficulties introduced by the nonlinear constraint can be eliminated by analyzing an associated penalized system. This is system is analyzed in detail in Section 3.

Recently, the author and others have examined simplifications of the Ericksen-Leslie equations that retain the original constraint. These works have still simplified equations and primarily examine a strongly coupled system of the Navier-Stokes equations and the transported heat flow of harmonic maps. This system, and its differences from the penalized system are discussed the Sections 4 and 5. The latter provides an outline of the recent work by the author and C.Y. Wang on local well-posedness [22].

We record open problems for both the penalized and non-penalized simplifications of the Ericksen-Leslie equations in both analysis and numerical analysis. Each of these problem should be viewed as a part of the over-arching problem of constructing a well-posedness theory for the Ericksen-Leslie equations. For other directions and problems, the interested reader is encourage to consult the surveys by Lin and Liu [44] and Lin [42].

Throughout this survey we will consider functions $\mathbf{u} : \Omega \to \mathbb{R}^3$ (usually fluid velocity field) and $\mathbf{d} : \Omega \to \mathbb{S}^2$ (usually the director or orientation field) for $\Omega \subset \mathbb{R}^3$ and where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$. We use the convention that upper indices represent components and lower indices represent derivatives. We also use the Einstein summation convention throughout – that is, we sum implicitly over any repeated index. Components are often cumbersome and hence we will often use the common operators from tensor calculus to provide concise expressions – namely, we use $\cdot, :$ for the vector and tensor inner-products respectively, and $\wedge$ for the outer or cross product of vectors or tensors.

2. Ericksen-Leslie equations

The Ericksen-Leslie equations for the hydrodynamic flow of nematic liquid crystals can be derived from the following conservation and balance laws in the way that the Navier-Stokes equations is often derived (for a derivation of the incompressible Navier-Stokes see [48], [10]). The Ericksen-Leslie equations were derived as result of extending the earlier hydrostatic theory for liquid crystals given by Oseen, Zocher, and Frank. In 1961, Ericksen [13] provide conservation laws for the hydrodynamic system for nematic liquid crystals. The system was completed by Leslie [35] in 1968 with the addition of constitutive relations. For our development we assume the following conservation and balance laws:

(1) *conservation of mass*
$$\frac{D}{Dt}\int_{\Omega}\rho dV = 0; \tag{2.1}$$
(2) *balance of linear momentum*
$$\frac{D}{Dt}\int_{\Omega}\rho\mathbf{u}dV = \int_{\Omega}\rho\mathbf{f}_{\text{body}}dV + \int_{\partial\Omega}\mathbf{f}_{\text{surf}}dS; \tag{2.2}$$
(3) *balance of angular momentum*
$$\frac{D}{Dt}\int_{\Omega}\rho(\mathbf{x}\wedge\mathbf{u})dV = \int_{\Omega}\rho(\mathbf{x}\wedge\mathbf{f}_{\text{body}}+\mathbf{k})dV + \int_{\partial\Omega}(\mathbf{x}\wedge\mathbf{f}_{\text{surf}}+\mathbf{l})dS. \tag{2.3}$$

Where, $\Omega\subset\mathbb{R}^3$, $\frac{D}{Dt}$ is material or convective time derivative, and

$$\begin{gathered}\rho = \text{ density},\quad \mathbf{x} = \text{ position},\\ \mathbf{u} = \text{ fluid velocity},\\ \mathbf{f}_{\text{body}} = \text{ external body force},\\ \mathbf{f}_{\text{surf}} = \text{ surface force (stress)},\\ \mathbf{k} = \text{ external body moment, and}\\ \mathbf{l} = \text{ surface moment.}\end{gathered}$$

If ρ is constant, the Reynolds transport theorem implies that
$$\nabla\cdot\mathbf{u} = 0 \quad \text{(incompressibility)}. \tag{2.4}$$
The application of the Reynolds transport theorem in the remaining integral balance laws yields:
$$\rho\frac{Du^i}{Dt} = \rho f^i_{\text{body}} + \nabla_j F^{ij} \quad \text{(point-wise balance of linear momentum)} \tag{2.5}$$
$$\rho K^i + \epsilon^{ijk}F^{kj} + \nabla_j L^{ij} = 0 \quad \text{(point-wise balance of angular momentum)} \tag{2.6}$$
where
$$f^i_{\text{surf}} = F^{ij}\nu^j \text{ and } l^i = L^{ij}\nu^j,\quad \nu = \text{normal to } \partial\Omega$$
and $\mathbf{F} = F^{ij}$ is the *stress tensor* and $\mathbf{L} = L^{ij}$ is the *couple stress tensor*.

The equations giving mass conservation (incompressiblity), balance of linear momentum, and balance of angular momentum can be thought of as kinematic equations (equations (2.4)-(2.6) respectively). That is, they are equations that describe motion, but they do not explain the origin of the forces generating the motion. Dynamic equations would also include the constitutive relations for the material that describe the origins of forces. One constitutive hypothesis that can be made for the derivation of the Ericksen-Leslie equations is a *rate of work* assumption. Namely,

$$\begin{aligned} &\int_\Omega \rho(\mathbf{f}_{\text{body}} \cdot \mathbf{u} + \mathbf{k} \cdot \hat{\mathbf{w}})dV + \int_{\partial\Omega} (\mathbf{f}_{\text{surf}} \cdot \mathbf{u} + \mathbf{l} \cdot \hat{\mathbf{w}})dS \\ &= \frac{D}{Dt} \int_\Omega \Big(\frac{1}{2}\rho\mathbf{u} \cdot \mathbf{u} + \sigma_F\Big)dV + \int_\Omega \mathcal{D}dV \end{aligned} \tag{2.7}$$

where $\mathcal{D}$ is the rate of viscous dissipation per unit volume, σ_F is the Oseen-Zocher-Frank elastic energy density, and $\hat{\mathbf{w}}$ is the axial vector (representing *local angular velocity*). The Oseen-Zocher-Frank elastic energy density is assumed to be *frame-indifferent*, *material-symmetric*, *even*, and *positive definite* (see [54] or [51]). These assumptions lead to *Frank's Formula* (see [54]) for σ_F namely:

$$\begin{aligned} \sigma_F(\mathbf{d}, \nabla\mathbf{d}) &= k_1(\operatorname{div}\mathbf{d})^2 + k_2(\mathbf{d} \cdot \operatorname{curl}\mathbf{d})^2 + k_3|\mathbf{d} \wedge \operatorname{curl}\mathbf{d}|^2 \\ &\quad + (k_2 + k_4)(\operatorname{tr}(\nabla\mathbf{d})^2 - (\operatorname{div}\mathbf{d})^2). \end{aligned} \tag{2.8}$$

Using the point-wise balance of linear momentum (2.5) and point-wise balance of angular momentum (2.6) one may simplify the rate of work assumption (2.7) through the application of the Reynolds transport theorem to find the point-wise relation:

$$F^{ij}\nabla_j u^j + L^{ij}\nabla_j w^i - w^i\epsilon^{ijk}F_s^{kj} = \frac{D\sigma_F}{Dt} + \mathcal{D}. \tag{2.9}$$

Here, $\mathbf{w} = \hat{\mathbf{w}} - \frac{1}{2}\nabla \wedge \mathbf{u}$ is the *relative angular velocity.*

What remains is to find specific forms of the stress tensor F_s^{ij} and the couple stress tensor L^{ij}. This is done using Ericksen's identity [13]:

$$\epsilon^{ijk}\Big(d^j\frac{\partial\sigma_F}{\partial d^k} + \nabla_p d^j\frac{\partial\sigma_F}{\partial\nabla_p d^k} + \nabla_p d^j\frac{\partial\sigma_F}{\partial\nabla_k d^p}\Big) = 0. \tag{2.10}$$

One finds that the stress tensor F^{ij} and the couple stress tensor L^{ij} may take the forms

$$\begin{gathered} F^{ij} = -p\delta^{ij} - \frac{\partial\sigma_F}{\partial\nabla_j d^p}\nabla_i d^p + \tilde{F}^{ij} \\ L^{ij} = \epsilon^{iqp}d^q\frac{\partial\sigma_F}{\partial\nabla_j d^p} + \tilde{L}^{ij}. \end{gathered} \tag{2.11}$$

Here, p is the pressure arising from incompressibility and $\tilde{F}^{ij}$ and $\tilde{L}^{ij}$ are the possible dynamic contributions. Thus, one has the following expression for $\mathcal{D}$:

$$\tilde{F}^{ij}\nabla_j u^i + \tilde{L}^{ij}\nabla_j\hat{w}^i + \hat{w}^i\epsilon^{ijk}\tilde{F}^{kj} = \mathcal{D}. \tag{2.12}$$

To find constitutive relations for the dynamic terms $\tilde{F}^{ij}$, $\tilde{L}^{ij}$ it is assumed that the dissipation $\mathcal{D}$ is positive. That is,

$$\tilde{F}^{ij}\nabla_j u^i + \tilde{L}^{ij}\nabla_j\hat{w}^i + \hat{w}^i\epsilon^{ijk}\tilde{F}^{kj} = \mathcal{D} \geq 0.$$

This assumption along with material-frame indifference, nematic symmetry, and linear dependence on d^i, N^i and S^{ij} leads to

$$\begin{gathered} \tilde{L}^{ij} = 0 \\ \tilde{F}^{ij} = \alpha_1 d^k S^{kp} d^p d^i d^j + \alpha_2 N^i d^j + \alpha_3 d^i N^j + \alpha_4 S^{ij} + \alpha_5 d^j S^{ik} d^k + \alpha_6 d^i S^{jk} d^k. \end{gathered} \tag{2.13}$$

Where $\alpha_1, \dots, \alpha_6$ are the Leslie viscosities and

$$S^{ij} = \frac{1}{2}(u^i_j + u^j_i) = \text{ rate of strain tensor},$$

$$W^{ij} = \frac{1}{2}(u^i_j - u^j_i) = \text{ vorticity tensor,}$$
$$N^i = \frac{Dd^i}{Dt} - W^{ij}d^j = \text{ co-rotational time flux of the director } \mathbf{d}.$$

Further manipulations, including a second application of Ericksen's identity, and the fact that $d^i d^i_j = 0$ lead to the full (isothermal) Ericksen-Leslie equations that we summarize now.

Summary of the Ericksen-Leslie equations: We may summarize the Ericksen-Leslie equations in component form as

$$\begin{gathered}
\rho\frac{Du^i}{Dt} = \rho F^i - (p+\sigma_F)_i + \tilde{g}^j d^j_i + G_j d^j_i + \tilde{F}^{ij}_j \\
u^i_i = 0 \\
\Big(\frac{\partial \sigma_F}{\partial d^i_j}\Big)_j - \frac{\partial \sigma_F}{\partial d^i} + \tilde{g}^i + G^i = \lambda d^i \\
d^i d^i = 1.
\end{gathered} \tag{2.14}$$

with constitutive relations for the viscous/dynamic stress tensor $\tilde{F}^{ij}$ and vector $\tilde{g}^i$ given by

$$\begin{gathered}
\tilde{t}^{ij} = \alpha_1 d^k S^{kp} d^p d^i d^j + \alpha_2 N^i d^j + \alpha_3 d^i N^j + \alpha_4 S^{ij} + \alpha_5 d^j S^{ik} d^k + \alpha_6 d^i S^{jk} d^k \\
\tilde{g}^i = -\gamma_1 N^i - \gamma_2 A^{ip} d^p.
\end{gathered} \tag{2.15}$$

and

$$\begin{gathered}
f^i_{\text{body}} = \text{external body for per unit mass,} \\
G^i = \text{generalized body force,} \\
\rho = \text{density,} \quad p = \text{pressure,} \\
\sigma_F = \text{Oseen-Zocher-Frank elastic energy density for nematics,} \\
\lambda = \text{Lagrange multiplier corresponding to the constraint } d^i d^i = 1, \\
\gamma_1 = \alpha_3 - \alpha_2 \geq 0 \\
\gamma_2 = \alpha_3 + \alpha_2 = \alpha_6 - \alpha_5 = \text{Parodi's relation,} \\
S^{ij} = \frac{1}{2}(u^i_j + u^j_i) = \text{ rate of strain tensor,} \\
W^{ij} = \frac{1}{2}(u^i_j - u^j_i) = \text{ vorticity tensor,} \\
N^i = \frac{Dd^i}{Dt} - W^{ij}d^j = \text{ co-rotational time flux of the director } \mathbf{d}.
\end{gathered} \tag{2.16}$$

3. Penalized simplifications of the Ericksen-Leslie equations

Intuitively, based upon the similarities between isotropic fluid flow modeled by the Navier-Stokes equation and the Ericksen-Leslie equations, early simplifications considered the Ericksen-Leslie equations as a system where the microscopic alignment is governed by the transported heat flow of harmonic maps into the sphere and the macroscopic fluid behavior is governed by the Navier-Stokes equations. This intuitive picture first appeared in the survey by Lin [40] and was in explored in detail by Lin and Liu in [38]. Lin and Liu in [38] gave a particular simplification of the Ericksen-Leslie equations inspired by the then recent work of Chen and Struwe [9], [8] on the heat flow harmonic maps. They provided a formulation of the Ericksen-Leslie equations that avoided the constraint that the director must have unit length. Specifically they studied the system

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{u} - \nabla\cdot(p\mathbb{I} + \lambda\nabla\mathbf{d}\odot\nabla\mathbf{d}) \tag{3.1}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{3.2}$$

$$\mathbf{d}_t + (\mathbf{u}\cdot\nabla)\mathbf{d} = \gamma(\Delta\mathbf{d} - f(\mathbf{d})). \tag{3.3}$$

in $\Omega\times(0,\infty)$ for $\Omega\subset\mathbb{R}^m$, $m = 2,3$. Here $\nabla\mathbf{d}\odot\nabla\mathbf{d} = d^i_j d^i_k$. To arrive at this system:

(1) The elastic constants in Oseen-Zocher-Frank elastic energy density are set as $k_1 = k_2 = k_3 = 1$ and $k_4 = 0$. This reduces the Oseen-Zocher-Frank elastic energy to the simpler Dirichlet energy $\frac{1}{2}\int_\Omega |\nabla\mathbf{d}|^2$
(2) The sphere constraint $d^i d^i = 1$ is handled via the penalization $F(\mathbf{d}) = \frac{1}{4\epsilon^2}(|\mathbf{d}|^2-1)^2$. That is, the Oseen-Zocher-Frank elastic energy is replaced by the functional $\int_\Omega \frac{1}{2}|\nabla\mathbf{d}|^2 + \frac{1}{4\epsilon^2}(|\mathbf{d}|^2-1)^2$.
(3) It is assumed for Leslie viscosities $\alpha_1,\dots,\alpha_6$ in (2.15) that $\alpha_1 = \alpha_5 = \alpha_6 = 0$. Hence the Ericksen-Leslie equations are reduced to equations having only three constants ν, γ, and λ
(4) Body force terms $\mathbf{f}_{\text{body}}$, G^i and other terms are removed.

For the system (3.1)-(3.3), Lin and Liu demonstrated in [38] the following:

Theorem 3.1 (Existence global weak solutions)**.** *Under the assumptions that* $\mathbf{u}_0(x)\in L^2(\Omega)$ *and* $\mathbf{d}_0(x)\in H^1(\Omega)$ *with* $\mathbf{d}_0|_{\partial\Omega}\in H^{3/2}(\partial\Omega)$*, the system* (3.1)-(3.3) *has a global weak solution* $(\mathbf{u},\mathbf{d})$ *such that*

$$\mathbf{u}\in L^2(0,T,H^1(\Omega))\cap L^\infty(0,T,L^2(\Omega))$$
$$\mathbf{d}\in L^2(0,T,H^2(\Omega))\cap L^\infty(0,T,H^1(\Omega))$$

for all $T\in(0,\infty)$*.*

Theorem 3.2 (Wellposedness for large viscosity)**.** *Problem* (3.1)-(3.3) *has a unique global classical solution* $(\mathbf{u},\mathbf{d})$ *provided that* $\mathbf{u}_0(x)\in H^1(\Omega)$*,* $\mathbf{d}_0(x)\in H^2(\Omega)$*,* $\dim(\Omega) = 2,3$*, and* $\nu\ge\nu_0(\lambda,\gamma,\mathbf{u}_0,\mathbf{d}_0)$*.*

Many others have studied the system (3.1)- (3.3). Some of the major contributions have been the following:

(1) **Partial Regularity.** In [39], Lin and Liu also demonstrated partial regularity of suitable weak solutions to (3.1)-(3.3) in $\Omega\subset\mathbb{R}^3$. This extended the work of Caffarelli, Kohn, and Nirenberg on the partial regularity of suitable weak solutions for the Navier-Stokes equations (see [6]). In particular, Lin and Liu were able to show that for sufficiently smooth initial and boundary data that there exists a suitable weak solution such that the singular set of this solution has one-dimensional parabolic Hausdorff measure zero. Specifically, a *suitable weak solution* $(\mathbf{u},\mathbf{d})$ to (3.1)- (3.3) in an open set $D\subset\mathbb{R}^3\times\mathbb{R}$ i. satisfies (3.1)- (3.3) weakly, ii. $(\mathbf{u},\mathbf{d})$ has the bounds

$$\int_{D\cap(\mathbb{R}^3\times t)} (|\mathbf{u}|^2 + |\nabla\mathbf{d}|^2 + F(\mathbf{d}))dx < E_1 = \text{constant}$$

$$\int\int_D (|\mathbf{u}|^2 + |\Delta\mathbf{d} - f(\mathbf{d})|) dxdt < E_2 = \text{constant}$$

and, iii. $(\mathbf{u}, \mathbf{d})$ satisfies the *generalized energy inequality*

$$\begin{aligned} &2\int_0^T \int_\Omega (|\nabla\mathbf{u}|^2 + |\nabla^2\mathbf{d}|^2)\phi dxdt \\ &\le \int_0^T \int_\Omega (|\mathbf{u}|^2 + |\nabla u|^2)(\phi_t + \Delta\phi) dxdt \\ &\quad + \int_0^T \int_\Omega (|\mathbf{u}|^2 + |\nabla u|^2 + 2P)\mathbf{u}\cdot\nabla\phi dxdt \\ &\quad + \int_0^T \int_\Omega ((\mathbf{u}\cdot\nabla)\mathbf{d})\cdot\nabla\phi dxdt + \int_0^T \int_\Omega \nabla f(\mathbf{d})\nabla\mathbf{d}\phi dxdt. \end{aligned} \tag{3.4}$$

(2) **Well-posedness of penalized Ericksen-Leslie equations.** Lin and Liu also examined the existence of global weak solutions to the Ericksen-Leslie equations in the late 1990s in [41]. Their main result was that under assumption $\mathbf{u}_0 \in L^2(\Omega)$ and $\mathbf{d}_0 \in H^1(\Omega)$ with $\mathbf{d}_0|_{\partial\Omega} \in H^{3/2}(\partial\Omega)$, the initial and boundary-value problem consisting of a penalized version of (2.14); that is,

$$\begin{gathered} \rho\frac{Du^i}{Dt} = -(p - \sigma_{LL})_i + \tilde{F}_j^{ij} \\ u_i^i = 0 \\ \Big(\frac{\partial\sigma_F}{\partial d_j^i}\Big)_j - \frac{\partial\sigma_F}{\partial d^i} + \tilde{g}^i = 0 \end{gathered} \tag{3.5}$$

Here $d^i d^i = 1$ has been excluded, and the energy density must be modified is modified. For example, $\sigma_F = \frac{1}{2}|\nabla d|^2 + \frac{1}{4\epsilon^2}(1-|d|^2)^2$. The system (3.5) retains many of the structures present in (2.14). Lin and Liu showed under certain assumptions that the system (3.5) has a global weak solution $(\mathbf{u}, \mathbf{d})$ such that

$$\begin{gathered} \mathbf{u} \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \\ \mathbf{d} \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \end{gathered}$$

for all $T \in (0,\infty)$. To achieve this result Lin and Liu required the following assumptions on the Leslie viscosities to insure control of the norm of $\mathbf{d}$:

$$\alpha_1, \alpha_4, \alpha_5 + \alpha_6 > 0,\ \gamma_1 < 0,\ \text{and } \gamma_2 = 0. \tag{3.6}$$

This simplification, as indicated by the authors, may have no physical meaning. Similar equations for the hydrodynamic flow of nematic liquid crystals were studied using different techniques by Coutand and Shkoller [12].

Very recently, Wu, Xu, and Liu in [59] continued the investigation of the well-posedness of the penalized Ericksen-Leslie system. They were able to examine the general penalized Ericksen-Leslie system with out un-natural assumption on the Leslie viscosities, namely, $\gamma_2 = 0$. Their main result [59, Theorem 4.1] states that the system (3.5) admits a unique global solution for larger viscosity and initial data $(\mathbf{u}_0, \mathbf{d}_0) \in V \times H^2$.

Other simplified models of the hydrodynamical flow of nematic liquid crystals along the lines of the Lin–Liu equations (3.1)- (3.3) have been proposed and analyzed. In the works [52, 58, 7], the authors considered

$$
\begin{gathered}
\nabla\cdot\mathbf{u}=0\\
\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}=\nabla\cdot\mathcal{T}+\mathcal{F}\\
\mathbf{d}_t+\mathbf{u}\cdot\nabla\mathbf{d}-\alpha\mathbf{d}\cdot\nabla\mathbf{u}+(1-\alpha)\mathbf{d}\cdot\nabla^T\mathbf{u}=\gamma(\Delta\mathbf{d}-\nabla_{\mathbf{d}}W(\mathbf{d}))
\end{gathered}
\tag{3.7}
$$

with

$$
\begin{gathered}
\mathcal{T}=\nu\mathcal{S}-\lambda\big[\nabla\mathbf{d}\odot\nabla\mathbf{d}+\alpha(\Delta\mathbf{d}-\nabla_{\mathbf{d}}W(\mathbf{d}))\otimes\mathbf{d}-(1-\alpha)\mathbf{d}\otimes(\Delta\mathbf{d}-\nabla_{\mathbf{d}}W(\mathbf{d}))\big]\\
\mathcal{S}=\nu(\nabla\mathbf{u}+\nabla^T\mathbf{u})
\end{gathered}
$$

for $(\mathbf{u},\mathbf{d},p):\Omega\times(0,T)\to\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}$ where $\mathbf{u}$ represents the velocity field, $\mathbf{d}$ represents the averaged macroscopic/continuum orientation, p is the hydrodynamic pressure. The constants ν, λ, and γ represent the viscosity, ratio of kinetic to potential energy, and the macroscopic relaxation time or Deborah number. The parameter $\alpha\in[0,1]$ is related to the shape of the liquid crystal molecule—$\alpha=1$ yields disc-like molecules, $\alpha=1/2$ yields spherical molecules, and $\alpha=1$ yields rod like molecules.

For this model, it is shown in [59] (Theorem 2.1) that under appropriate assumptions on initial data there exists global weak solutions with periodic boundary conditions. The authors of [7] have a similar result, but with appropriate Dirichlet and Neumann boundary data for the velocity field and the orientation field.

(3) **Limits of sequences of solutions to the penalized equations.** In [41] (Theorem 7.1) it was shown, up to a subsequence, solutions $(\mathbf{u}_\epsilon,\mathbf{d}_\epsilon)$ of the penalized Ericksen-Leslie equations converge strongly in $L^2(0,T;L^2(\Omega))\times L^2(0,T;H^1(\Omega))$ to the sharp system (2.14) with the addition of matrix defect measure appearing with the other stress terms. Whether this defect measure is identically zero is an outstanding and difficult open problem.

4. Non-penalized simplifications of the Ericksen-Leslie equations

Beginning in the late 1980s Hardt, Kinderlehrer and Lin started the mathematical analysis of static solutions to the Ericksen-Leslie equations or minimizers of the Oseen-Zocher-Frank elastic energy functional [19]. The equations for static solutions follow naturally from the Ericksen-Leslie equations by setting the velocity field $\mathbf{u}$ to zero. Such equations are the Euler-Lagrange equations of the Oseen-Zocher-Frank elastic energy functional ($\int_\Omega\sigma_F$). Hardt, Kinderlehrer, and Lin used direct methods to study the Oseen-Zocher-Frank elastic energy functional. They established existence and partial regularity of minimizers of the Oseen-Zocher-Frank functional. To be precise, the major results of their work are summarized in the following two theorems.

Theorem 4.1 (Existence of minimizers)**.** *For $\mathbf{d}_0:\partial\Omega\to\mathbb{S}^2$ a Lipshitz map, the admissible class of minimizers*

$$\mathcal{A}(\mathbf{d}_0):=\{\mathbf{d}\in H^1(\Omega,\mathbb{S}^2):\mathbf{d}_0=\text{ trace of }\mathbf{d}\text{ on }\partial\Omega\}$$

is non-empty. Furthermore, for any $\mathbf{d}_0:\partial\Omega\to\mathbb{S}^2$ there exists $\mathbf{d}\in\mathcal{A}(\mathbf{d}_0)$ such that

$$\mathcal{F}[\mathbf{d}]=\inf_{\mathbf{d}\in\mathcal{A}(\mathbf{d}_0)}\int_\Omega\sigma_F(\mathbf{d},\nabla\mathbf{d}).$$

Theorem 4.2 (Interior partial regularity). *If* $\mathbf{d} \in H^1(\Omega, \mathbb{S}^2)$ *is a minimizer of* $\mathcal{F}$*, then* $\mathbf{d}$ *is analytic on* $\Omega \setminus Z$ *for some relatively closed subset* Z *of* Ω *which has one-dimensional Hausdorff measure zero.*

The work of Hardt, Kinderlehrer and Lin [19] inspired many other mathematical works on the static theory of liquid crystals. Hardt and Lin expanded the study of defects of nematics in [20]. The authors Cohen [11], Kinderlehrer, Ou [31], and Hélein [29] examined the stability of static solutions. Lin, Luskin, Alouges, Ghidaglia, and others explored static solutions of nematic liquid crystals by numerically optimizing the Oseen-Zocher-Frank elastic energy functional (see [47], [1], [2]). There remain open problems for the static theory of liquid crystals, many of which revolve around better understanding of the nature of the Oseen-Zocher-Frank energy functional. Though, not the focus of this survey it worth mentioning a few very interesting problems for the static theory of liquid crystals.

(1) **Monotonicity of Oseen-Zocher-Frank functional.** To the best of the knowledge of the author there is no mathematical proof of monotonicity or example demonstrating the lack of monotonicity. That is, it is unknown that if $\mathbf{d}$ is a static solution (critical point of the Oseen-Zocher-Frank functional) whether one has

$$\int_{B_r(x_0)} \sigma_F(\mathbf{d}, \nabla \mathbf{d}) \le \int_{B_R(x_0)} \sigma_F(\mathbf{d}, \nabla \mathbf{d}), \ 0 < r \le R.$$

As mentioned in the survey [44] by F.H. Lin and C. Liu (without reference), there exist reliable computations that indicate that the Oseen-Zocher-Frank energy is monotone.

(2) **Numerical exploration of static solutions of nematic liquid crystals.** Beginning with the work Lin and Luskin [47] in the late 1980s many mathematicians have explored techniques for numerical solution of the harmonic map problem. Namely, Lin and Luskin explored harmonic maps into spheres via relaxation. Starting in the mid 1990s Alouges [1] presented a creative and efficient algorithm to find harmonic maps by minimization of the Dirichlet energy functional into spheres or convex targets. The algorithm transforms the numerical difficulty of constraining the range to a manifold (a nonlinear constraint) present in numerical solution of harmonic maps into a more tractable constraint on tangent space (a linear constraint). One of the key properties that allows the Alouges algorithm to work is the fact that projection onto the target (say the sphere) decreases the Dirichlet energy. The same is not true for the Oseen-Zocher-Frank functional. Alouges and Ghidaglia were able to deal with the more general Oseen-Zocher-Frank functional by constructing auxilliary functionals that matched the Oseen-Zocher-Frank functional when properly restricted. Alouges and Ghidaglia in [2] were unable to provide a proof of the convergence of the Alouges algorithm for the Oseen-Zocher-Frank functional. To the best of the knowledge of the author the program of Alouges and Ghidaglia remains unfinished.

More recently, Bartels re-imagined the Alouges algorithm in a finite element context [3]. Bartels found that there must be a restriction upon the finite element discretization in order to achieve the energy decreasing property. Bartels demonstrated that the use of *acute* finite elements yield an energy decreasing algorithm. The work [3] only explores the Dirichlet energy. A similar analysis of the more general Oseen-Zocher-Frank functional may be possible if suitable constraint on the discretization can be found.

Although, it is a natural transition to consider the dynamic non-penalized Ericksen-Leslie equations (2.14) such an analysis has only recently begun and is as result still incomplete. Here we discuss known results a non-penalized Lin–Liu type system (3.1)-(3.3). For this system there are fairly complete results for two dimensional domains and preliminary results in three dimensions. For the two-dimensional work, we discuss the work of Lin, Lin and Wang [43] and for three dimensions, we discuss the work of Hineman and Wang [22]. One should also see Hong [23] and Xu-Zhang [60] and Hong–Xin [24].

Lin, Lin, and Wang [43] analyzed the simplified Ericksen-Leslie system

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla P = -\lambda\nabla\cdot(\nabla\mathbf{d}\odot\mathbf{d}) \tag{4.1}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{4.2}$$

$$\mathbf{d}_t + \mathbf{u}\cdot\nabla\mathbf{d} = \gamma(\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}) \tag{4.3}$$

for $\mathbf{u}:\Omega\to\mathbb{R}^2$ and $\mathbf{d}:\Omega\to\mathbb{S}^2$ where $\Omega\subset\mathbb{R}^2$ and subject to

$$\mathbf{u}_0\in\mathbf{H}, \mathbf{d}_0\in H^1(\Omega,\mathbb{S}^2),\ \text{and } d_0\in C^{2,\beta}(\partial\Omega,\mathbb{S}^2) \text{ for some } \beta\in(0,1). \tag{4.4}$$

Where,

$$\begin{gathered}\mathbf{H} = \text{closure of } C_0^\infty(\Omega,\mathbb{R}^2)\cap\{\mathbf{u}:\nabla\cdot\mathbf{u}=0\} \text{ in } L^2(\Omega,\mathbb{R}^2),\\ \mathbf{J} = \text{closure of } C_0^\infty(\Omega,\mathbb{R}^2)\cap\{\mathbf{u}:\nabla\cdot\mathbf{u}=0\} \text{ in } H^1(\Omega,\mathbb{R}^2),\ \text{and}\\ H^1(\Omega,\mathbb{S}^2) = \{\mathbf{d}\in H^1(\Omega,\mathbb{R}^3):\mathbf{d}(x)\in\mathbb{S}^2 \text{ a.e. } x\in\Omega\}.\end{gathered}$$

They proved the following result.

Theorem 4.3 (Regularity). *For $0<T<+\infty$ assume*

$$\mathbf{u}\in L^\infty([0,T],\mathbf{H})\cap L^2([0,T],\mathbf{J})$$

and $\mathbf{d}\in L^2([0,T],H^1(\Omega,\mathbb{S}^2))$ is a suitable weak solution of (4.1)-(4.3) with (4.4). If in addition, $\mathbf{d}\in L^2([0,T],H^2(\Omega))$, then $(\mathbf{u},\mathbf{d})\in C^\infty(\Omega\times(0,T])\cap C^{2,1}_\beta(\bar\Omega)\times(0,T]$.

Theorem 4.4 (Global Weak Solutions of Partial Regularity). *For data satisfying (4.4), there exist global suitably weak solutions $\mathbf{u}\in L^\infty([0,\infty),\mathbf{H})\cap L^2([0,\infty),\mathbf{J})$ and $\mathbf{d}\in L^\infty([0,\infty),H^1(\Omega,\mathbb{S}^2))$ of the equations (4.1)-(4.3), which has the following properties:*

(1) There exists $L\in\mathbb{N}$ depending only on $(\mathbf{u}_0,\mathbf{d}_0)$ and $0<T_1<\dots<T_L$, $1\le i\le L$, such that

$$(\mathbf{u},\mathbf{d})\in C^\infty(\Omega\times((0,\infty)\setminus\{T_i\}_{i=1}^L))\cap C^{2,1}_\beta(\bar\Omega\times((0,\infty)\setminus\{T_i\}_{i=1}^L)).$$

(2) Each singular time T_i, $1\le i\le L$, can be characterized by

$$\liminf_{t\uparrow T_i}\max_{x\in\bar\Omega}\int_{\Omega\cap B_r(x)}(|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)(y,t)dy\ge 8\pi,\ \forall r>0.$$

Moreover, there exist $x_m^i\to x_0^i\in\Omega$, $t_i\uparrow T_i$, $r_m^i\downarrow 0$ and a non-constant smooth harmonic map $\omega_i:\mathbb{R}^2\to\mathbb{S}^2$ with finite energy such that as $m\to\infty$,

$$(\mathbf{u}_m^i,\mathbf{d}_m^i)\to(0,\omega_i) \text{ in } C^2_{\rm loc}(\mathbb{R}^2\times[\infty,0]),$$

where

$$\mathbf{u}_m^i(x,t) = r_m^i\mathbf{u}(x_m^i+r_m^ix,t_m^i+(r_m^i)^2t),\ \mathbf{d}_m^i(x,t)=\mathbf{d}(x_m^i+r_m^ix,t_m^i+(r_m^i)^2t).$$

(3) Set $T_0=0$. Then, for $0\le i\le L-1$,

$$|\mathbf{d}_t|+|\nabla^2\mathbf{d}|\in L^2(\Omega\times[T_i,T_{i+1}-\epsilon]),\ |\mathbf{u}_t|+|\nabla^2\mathbf{u}|\in L^{4/3}(\Omega\times[T_i,T_{i+1}-\epsilon])$$

for any $\epsilon > 0$, and for any $0 < T_L < T < \infty$,
$$|\mathbf{d}_t| + |\nabla^2 \mathbf{d}| \in L^2(\Omega \times [T_L, T]), \ |\mathbf{u}_t| + |\nabla^2 \mathbf{u}| \in L^{4/3}(\Omega \times [T_L, T]).$$

(4) There exist $t_k \uparrow \infty$ and a harmonic map $\mathbf{d}_\infty \in C^\infty(\Omega, \mathbb{S}^2) \cap C^{2,\beta}(\bar{\Omega}, \mathbb{S}^2)$ with $\mathbf{d}_\infty = \mathbf{d}_0$ on $\partial\Omega$ such that $\mathbf{u}(\cdot, t_k) \to 0$ in $H^1(\Omega)$, $\mathbf{d}(\cdot, t_k) \to \mathbf{d}_\infty$ weakly in $H^1(\Omega)$, and there exist $l \in \mathbb{N}$, points $\{x_i\}_{i=1}^l \in \Omega$, and $\{m_i\}_{i=1}^l \subset \mathbb{N}$ such that
$$|\nabla \mathbf{d}(\cdot, t_k)|^2 dx \to |\nabla \mathbf{d}_\infty|^2 dx + \sum_{i=1}^{l} 8\pi m_i \delta_{x_i}.$$

(5) If $(\mathbf{u}_0, \mathbf{d}_0)$ satisfies
$$\int_\Omega (|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2) \le 8\pi, \tag{4.5}$$
then $(\mathbf{u}, \mathbf{d}) \in C^\infty(\Omega \times (0, \infty)) \cap C^{2,1}_\beta(\bar{\Omega} \times (0, \infty))$. Moreover, there exist $t_k \uparrow \infty$ and $\mathbf{d}_\infty \in C^\infty(\Omega, \mathbb{S}^2) \cap C^{2,\beta}(\bar{\Omega}, \mathbb{S}^2)$ with $\mathbf{d}_\infty = d_0$ on $\partial\Omega$ such that $(\mathbf{u}(\cdot, t_k), \mathbf{d}(\cdot, t_k)) \to (0, \mathbf{d}_\infty)$ in $C^2(\Omega)$.

Though the system treated Lin, Lin and Wang in [43] is a greatly simplified version of the Ericksen-Leslie equations it preserves the sphere constraint on the director. This nonlinear constraint seems to make it impossible apply a Galerkin-type technique to the problem. To prove their main theorem on the local existence (Theorem 4.4) the authors of [43] approximated weak solutions with strong solutions that they obtained through fixed point iteration. The proof is completed by using energy inequalities and the regularity result Theorem 4.3 to determine the maximal singular time. This method was originally used by Leray in [34] to prove local existence (and uniqueness) of weak solutions to the Navier-Stokes equation. Many others have explored and extended the techniques of Leray [50, 16, 30, 15, 18, 48, 33].

The work Lin, Lin, and Wang in [43] did not include results on the uniqueness of weak solutions of (4.1)-(4.3) but indicated that they believed the solution to be unique. A proof the uniqueness of solution to (4.1)-(4.3) was given by Lin and Wang [45], specifically, they proved the following result.

Theorem 4.5. *For $0 < T \le +\infty$, $\mathbf{u}_0 \in L^2(\Omega, \mathbb{R}^2)$ with $\nabla \cdot \mathbf{u}_0 = 0$ and $\mathbf{d}_0 : \Omega \to \mathbb{S}^2$ with $\nabla \mathbf{d}_0 \in L^2(\Omega, \mathbb{R}^6)$, suppose that for $i = 1, 2$ $\mathbf{u}_i \in L^\infty_t L^2_x \cap L^2_t H^1_x(\Omega \times [0, T], \mathbb{R}^2)$, $\nabla P_i \in L^{4/3}_t L^{4/3}_x(\Omega \times [0, T], \mathbb{R}^2)$, and $\mathbf{d}_i \in L^\infty_t \dot{H}^1_x \cap L^2_t \dot{H}^2_x(\Omega \times [0, T], \mathbb{S}^2)$ are a pair of weak solution to (4.1)-(4.3) under either*

(1) *when $\Omega = \mathbb{R}^2$, the same initial condition:*
$$(\mathbf{u}_i, \mathbf{d}_i)|_{t=0} = (\mathbf{u}_0, \mathbf{d}_0), \ i = 1, 2,$$
(2) *or, when $\Omega \subset \mathbb{R}^2$ is a bounded domain, the same initial and boundary conditions:*
$$(\mathbf{u}_i, \mathbf{d}_i) = (u_0, d_0), \ \text{on } \Omega \times \{0\}, \ (\mathbf{u}_i, \mathbf{d}_i) = (0, \mathbf{d}_0), \ \text{on } \partial\Omega \times (0, T), \ i = 1, 2$$
with $\mathbf{d}_0 \in C^{2,\beta}(\partial\Omega, \mathbb{S}^2)$ for some $\beta \in (0, 1)$.

Then $(\mathbf{u}_1, \mathbf{d}_1) \equiv (\mathbf{u}_2, \mathbf{d}_2)$ in $\Omega \times [0, T)$.

An alternative method for uniqueness of solutions in Serrin's class [50] to (4.1)-(4.3) will be discussed in the forthcoming article by Huang and Wang [28]. This work builds upon the work of Huang and Wang on the uniqueness of heat flow of harmonic maps investigated in [27].

As this work is still very recent, there remain many open problems. In the two-dimensional case it would it would still benefit our understanding of the full Ericksen-Leslie system to re-introduce various terms into the simplified equations (4.1)-(4.3). The problems facing closer approximations of the non-penalized Ericksen-Leslie will require different tools than were employed for the penalized Ericksen-Leslie equations (3.5) described above.

5. New findings in three dimensions

Recently, the author and Wang considered the Cauchy problem for a simplified hydrodynamic system modeling the flow of nematic liquid crystal materials in $\mathbb{R}^3$ [21]. Specifically, they considered the system for $(\mathbf{u}, P, \mathbf{d}) : \mathbb{R}^3 \times [0, T) \to \mathbb{R}^3 \times \mathbb{R} \times \mathbb{S}^2$, $0 < T \le \infty$ given by

$$\begin{gathered} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla P = -\lambda \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \quad \text{in } \mathbb{R}^3 \times (0, T), \\ \nabla \cdot \mathbf{u} = 0, \quad \text{in } \mathbb{R}^3 \times (0, T), \\ \mathbf{d}_t + \mathbf{u} \cdot \nabla \mathbf{d} = \gamma (\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}), \quad \text{in } \mathbb{R}^3 \times (0, T), \\ (\mathbf{u}, \mathbf{d}) = (\mathbf{u}_0, \mathbf{d}_0), \quad \text{on } \mathbb{R}^3 \times \{0\} \end{gathered} \tag{5.1}$$

for a given initial data $(\mathbf{u}_0, \mathbf{d}_0) : \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{S}^2$ with $\nabla \cdot \mathbf{u}_0 = 0$. Here again $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$ represents the velocity field of the fluid, $\mathbf{d} : \mathbb{R}^3 \to \mathbb{S}^2$ is a unit vector field representing the macroscopic molecular orientation of the nematic liquid crystal material, $P : \mathbb{R}^3 \to \mathbb{R}$ represents the pressure function. The constants ν, λ, and γ are positive constants that represent the viscosity of the fluid, the competition between kinetic and potential energy, and the microscopic elastic relaxation time for the molecular orientation field.

Because of the super-critical nonlinear term $\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d})$ in $(5.1)_1$, it has been an outstanding open problem whether there exists a global Leray-Hopf type weak solution to (5.1) in $\mathbb{R}^3$ for any initial data $(\mathbf{u}_0, \mathbf{d}_0) \in L^2(\mathbb{R}^3, \mathbb{R}^3) \times \dot{W}^{1,2}(\mathbb{R}^3, S^2)$ with $\nabla \cdot \mathbf{u}_0 = 0$. That is, a solution to (5.1) for which one has:

1. $(\mathbf{u}, P, \mathbf{d})$ is a unique weak solution to (5.1).
2. $(\mathbf{u}, P, \mathbf{d})$ satisfies the natural energy inequality.

Preceding this work, it has been shown that in $\mathbb{R}^3$ there exists a local, unique, strong solution to (5.1) for any initial data $\mathbf{u}_0 \in W^{s,2}(\mathbb{R}^3)$ and $\mathbf{d}_0 \in W^{s+1,2}(\mathbb{R}^3, \mathbb{S}^2)$ for $s > 3$ with $\nabla \cdot \mathbf{u}_0 = 0$, see for example [57]. Huang–Wang in[26] established a blow up criterion for local strong solutions to (5.1), similar to the Beale-Kato-Majda criterion for the Navier-Stokes equation (see [5]). Li-Wang [36] obtained the global existence of strong solutions for to (5.1) with small initial data in certain Besov spaces. In [56], Wang obtained the global (or local) well-posedness of (5.1) for initial data $(\mathbf{u}_0, \mathbf{d}_0)$ belonging to the space $\mathrm{BMO}^{-1} \times \mathrm{BMO}$ with $\nabla \cdot \mathbf{u}_0 = 0$, which is a invariant space under parabolic scaling associated with (5.1), with small norms.

The main contribution of this work is the local well-posedness of (5.1) for any initial data $(\mathbf{u}_0, \mathbf{d}_0)$ such that $(\mathbf{u}_0, \nabla \mathbf{d}_0) \in L^3_{\mathrm{uloc}}(\mathbb{R}^3)$. Henceforth $L^3_{\mathrm{uloc}}(\mathbb{R}^3)$ denotes the space of uniformly locally L^3-integrable functions. It turns out that $L^3_{\mathrm{uloc}}(\mathbb{R}^3)$ is also invariant under parabolic scaling associated with (5.1). Similar scaling symmetry for the Navier-Stokes equation has inspired exploration of the Cauchy problem with initial data in various L^p spaces (see for example [50, 6, 30, 15]). Uniform local spaces space have also been employed in a number of natural ways in the analysis of the Navier-Stokes equation (see [4, 33]).

Now we give the definition of $L^3_{\rm uloc}(\mathbb{R}^3)$. The readers can consult the monograph by Lemarié-Rieusset [33] for applications of the space $L^3_{\rm uloc}(\mathbb{R}^3)$ to the Navier-Stokes equation.

Definition 5.1. A function $f\in L^3_{\rm loc}(\mathbb{R}^3)$ belongs to the space $L^3_{\rm uloc}(\mathbb{R}^3)$ consisting of uniformly locally L^3-integrable functions, if there exists $0<R<+\infty$ such that

$$\|f\|_{L^3_R(\mathbb{R}^3)} := \sup_{x\in\mathbb{R}^3}\Big(\int_{B_R(x)}|f|^3\Big)^{1/3}<+\infty. \tag{5.2}$$

For an open set $U\subset\mathbb{R}^3$, $f\in L^3_{\rm uloc}(U)$ if $f\chi_U\in L^3_{\rm uloc}(\mathbb{R}^3)$, here χ_U is the characteristic function of U.

It is clear that

- $L^3(\mathbb{R}^3)\subset L^3_{\rm uloc}(\mathbb{R}^3)$.
- If $f\in L^3_{\rm uloc}(\mathbb{R}^3)$, then $\|f\|_{L^3_R(\mathbb{R}^3)}$ is finite for any $0<R<+\infty$. For any two $0<R_1\le R_2<\infty$, it holds

$$\|f\|_{L^3_{R_1}(\mathbb{R}^3)}\le \|f\|_{L^3_{R_2}(\mathbb{R}^3)}\lesssim \Big(\frac{R_2}{R_1}\Big)\|f\|_{L^3_{R_1}(\mathbb{R}^3)},\quad \forall f\in L^3_{\rm uloc}(\mathbb{R}^3). \tag{5.3}$$

- $L^3_{\rm uloc}(\mathbb{R}^3)\subset \cap_{0<R<\infty}{\rm BMO}^{-1}_R(\mathbb{R}^3)$ (see [32] or [56]). Moreover, for any $0<R<\infty$, it holds

$$[f]_{{\rm BMO}^{-1}_R(\mathbb{R}^3)}\lesssim \|f\|_{L^3_R(\mathbb{R}^3)},\quad \forall\ f\in L^3_{\rm uloc}(\mathbb{R}^3). \tag{5.4}$$

Theorem 5.2. *There exist $\epsilon_0>0$ and $\tau_0>0$ such that if $\mathbf{u}_0:\mathbb{R}^3\to\mathbb{R}^3$, with $\nabla\cdot\mathbf{u}_0=0$, and $\mathbf{d}_0:\mathbb{R}^3\to\mathbb{S}^2$ satisfies, for some $0<R<+\infty$ and $e_0\in S^2$,*

$$\begin{gathered}|||(\mathbf{u}_0,\nabla\mathbf{d}_0)|||_{L^3_R(R^3)} := \sup_{x\in\mathbb{R}^3}\Big(\int_{B_R(x)}|\mathbf{u}_0|^3+|\nabla\mathbf{d}_0|^3\Big)^{1/3}\le\epsilon_0,\\ \lim_{K\uparrow\infty}\|\mathbf{d}_0-e_0\|_{L^3_R(\mathbb{R}^3\setminus B_K)}\le\epsilon_0,\end{gathered} \tag{5.5}$$

then there exist $T_0\ge\tau_0R^2$ and a unique solution $(\mathbf{u},\mathbf{d}):\mathbb{R}^3\times[0,T_0)\to\mathbb{R}^3\times\mathbb{R}\times\mathbb{S}^2$ of (5.1) *that enjoys the following properties:*
(i) For $t\downarrow 0$, $(\mathbf{u}(t),\mathbf{d}(t))\to(\mathbf{u}_0,\mathbf{d}_0)$ and $\nabla\mathbf{d}(t)\to\nabla\mathbf{d}_0$ in $L^3_{\rm loc}(\mathbb{R}^3)$.
(ii)

$$\begin{gathered}(\mathbf{u},\mathbf{d})\in\cap_{0<\delta<T_0}C^\infty_b(\mathbb{R}^3\times[\delta,T_0-\delta],\mathbb{R}^3\times\mathbb{S}^2),\\ (\mathbf{u},\nabla\mathbf{d})\in\cap_{0<T'<T_0}C^0_*([0,T'],L^3_{\rm uloc}(\mathbb{R}^3)).\end{gathered}$$

(iii)

$$|||(\mathbf{u}(t),\nabla\mathbf{d}(t))|||_{L^\infty([0,\tau_0R^2],L^3_R(\mathbb{R}^3))}\le C\epsilon_0. \tag{5.6}$$

(iv) If $T_0<+\infty$ is the maximum time interval then it must hold

$$\limsup_{t\uparrow T_0}|||(\mathbf{u}(t),\nabla\mathbf{d}(t))|||_{L^3_r(\mathbb{R}^3)}>\epsilon_0,\quad\forall\ 0<r<\infty. \tag{5.7}$$

Remark 5.3. The ideas to prove Theorem 5.2 are motivated by those employed by [43]. There are six main ingredients:

(1) Approximate the initial data $(\mathbf{u}_0, \mathbf{d}_0)$ strongly in $L^p_{\rm loc} \cap W^{1,p}_{\rm loc}(\mathbb{R}^3)$ by a sequence of smooth initial data $\{(\mathbf{u}_0^k, \mathbf{d}_0^k)\} \subset C^\infty(\mathbb{R}^3, \mathbb{R}^3 \times \mathbb{S}^2) \cap (L^p(\mathbb{R}^3, \mathbb{R}^3) \times \dot{W}^{1,p}(\mathbb{R}^3, \mathbb{S}^2))$ for $p = 2, 3$, with $\nabla \cdot \mathbf{u}_0^k = 0$ and $\|(\mathbf{u}_0^k, \nabla \mathbf{d}_0^k)\|_{L^3_1(\mathbb{R}^3)}$ uniformly small for all k (see Lemma 5.7 below). We would like to point out that the existence of such an sequence is highly nontrivial.
(2) Apply the fixed-point argument, similar to that by Lin-Lin-Wang [43], to obtain a sequence of smooth solutions $(\mathbf{u}^k, P^k, \mathbf{d}^k)$ of (5.1) in $\mathbb{R}^3 \times [0, T_k]$ for $0 < T_k < +\infty$, with the initial data $(\mathbf{u}_0^k, \mathbf{d}_0^k)$, such that $(\mathbf{u}^k, \mathbf{d}^k) \in C^\infty(\mathbb{R}^3 \times [0, T_k], \mathbb{R}^3 \times \mathbb{S}^2) \cap C([0, T_k], L^2(\mathbb{R}^3) \times \dot{W}^{1,2}(\mathbb{R}^3))$.
(3) Utilize the local L^3-energy inequality (5.14) for $(\mathbf{u}^k, P^k, \mathbf{d}^k)$ to show that there is a uniform lower bound $\tau_0 > 0$ for T_k such that $\sup_{0 \le t \le \tau_0} \|(\mathbf{u}^k(t), \nabla \mathbf{d}^k(t))\|_{L^3_1(\mathbb{R}^3)}$ is uniformly small for all k. It turns out that $(\mathbf{u}^k, \nabla \mathbf{d}^k) \in C([0, T_k], L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3))$ plays an important role in the estimate of P^k in $L^{3/2}_{\rm loc}(\mathbb{R}^3)$.
(4) Apply the ϵ_0-regularity Theorem 5.14, which is analogous to that by Caffarelli-Kohn-Nirenberg [6] on the Navier-Stokes equation, to obtain a priori derivative estimates of $(\mathbf{u}^k, \mathbf{d}^k)$ in $\mathbb{R}^3 \times [0, \tau_0]$. In order to handle the nonlinearity $\nabla \cdot (\nabla \mathbf{d}^k \odot \nabla \mathbf{d}^k)$ in $(5.1)_1$, we do assume an additional smallness condition of $\|\nabla \mathbf{d}^k\|_{L^\infty_t L^3_x(P_1)}$ in Theorem 5.14. Then take a limit of $(\mathbf{u}^k, P^k, \mathbf{d}^k)$ to obtain the local existence of $L^3_{\rm uloc}$-solution $(\mathbf{u}, P, \mathbf{d})$ of (5.1) in $\mathbb{R}^3 \times [0, \tau_0]$.
(5) Apply Theorem 5.14 again to characterize the finite maximal time interval T_0 of $(\mathbf{u}, P, \mathbf{d})$.
(6) Adapt the proof of [56] to show that the uniqueness holds for $L^3_{\rm uloc}$-solutions that satisfy the same properties as the solution $(\mathbf{u}, P, \mathbf{d})$ constructed as above.

Remark 5.4. Since the exact values of ν, λ, γ do not play a role in the proof of Theorem 5.2, we may henceforth assume $\nu = \lambda = \gamma = 1$.

For a solution $(\mathbf{u}, P, \mathbf{d})$ to (5.1), denote its L^3-energy by

$$E_3(u, \nabla \mathbf{d})(t) = \int_{\mathbb{R}^3} (|\mathbf{u}(t)|^3 + |\nabla \mathbf{d}(t)|^3), \quad t \ge 0.$$

Concerning the global well-posedness of (5.1), we have the following result.

Theorem 5.5. *There exists an $\epsilon_0 > 0$ such that if*

$$(\mathbf{u}_0, \mathbf{d}_0) \in L^3(\mathbb{R}^3, \mathbb{R}^3) \times \dot{W}^{1,3}(\mathbb{R}^3, \mathbb{S}^2),$$

with $\nabla \cdot \mathbf{u}_0 = 0$, satisfies

$$E_3(\mathbf{u}_0, \nabla \mathbf{d}_0) \le \epsilon_0^3, \tag{5.8}$$

then there exists a unique global solution $(\mathbf{u}, \mathbf{d}) : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3 \times \mathbb{R} \times \mathbb{S}^2$ of (5.1) such that $(\mathbf{u}, \mathbf{d}) \in C^\infty(\mathbb{R}^3 \times (0, +\infty)) \cap C([0, \infty), L^3(\mathbb{R}^3) \times \dot{W}^{1,3}(\mathbb{R}^3))$, $E_3(\mathbf{u}, \nabla \mathbf{d})(t)$ is monotone decreasing for $t \ge 0$, and

$$\|\nabla^m \mathbf{u}(t)\|_{L^\infty(\mathbb{R}^3)} + \|\nabla^{m+1} \mathbf{d}(t)\|_{L^\infty(\mathbb{R}^3)} \le \frac{C\epsilon_0}{t^{\frac{m}{2}}}, \quad \forall\ t > 0,\ m \ge 0. \tag{5.9}$$

Remark 5.6. The first conclusion of Theorem 5.5 has already been proven by Lin [46], based on refinement of the argument by Wang in [56].

In the remainder of this section we discuss some of the details of the proof of Theorem 5.2 outlined in Remark 5.3. For complete details the reader is encouraged to consult the work of Hineman and Wang [22].

Step 1: Approximation. The first two ingredients listed in the Remark 5.3 involve the approximation of rough initial data by smooth initial data. This is precisely

Lemma 5.7. *For a sufficiently small* $\epsilon_0 > 0$, *let* $(\mathbf{u}_0, \mathbf{d}_0) : \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{S}^2$, *with divergence free* $\mathbf{u}_0 \in L^3_{\mathrm{uloc}}(\mathbb{R}^3)$, *satisfy, for some* $e_0 \in \mathbb{S}^2$,

$$|||(\mathbf{u}_0, \nabla \mathbf{d}_0)|||_{L^3_1(\mathbb{R}^3)} \le \epsilon_0 \text{ and } \lim_{K\to\infty} \|\mathbf{d}_0 - e_0\|_{L^3_1(\mathbb{R}^3\setminus B_K)} \le \epsilon_0, \tag{5.10}$$

then there exist

$$\{(\mathbf{u}^k_0, \mathbf{d}^k_0)\} \subset C^\infty(\mathbb{R}^3, \mathbb{R}^3 \times \mathbb{S}^2) \cap \cap_{p=2}^3 (L^p(\mathbb{R}^3, \mathbb{R}^3) \times \dot{W}^{1,p}(\mathbb{R}^3, \mathbb{S}^2))$$

such that the following properties hold:
(i) $\nabla \cdot \mathbf{u}^k_0 = 0$ *in* $\mathbb{R}^3$ *for all* $k \ge 1$.
(ii) As $k \to \infty$,

$$(\mathbf{u}^k_0, \mathbf{d}^k_0) \to (\mathbf{u}_0, \mathbf{d}_0) \text{ and } \nabla \mathbf{d}^k_0 \to \nabla \mathbf{d}_0 \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}^3) \text{ for } p = 2, 3. \tag{5.11}$$

(iii) There exist $C_0 > 0$ *and* $k_0 > 1$ *such that for any* $k \ge k_0$,

$$|||(\mathbf{u}^k_0, \nabla \mathbf{d}^k_0)|||_{L^3_1(\mathbb{R}^3)} \le C_0 \epsilon_0. \tag{5.12}$$

Remark 5.8. The proof of Lemma 5.7 is non-trivial and technical. It naturally requires separate approximations for velocity initial data $\mathbf{u}_0$ and the director initial data $\mathbf{d}_0$. The approximation of $\mathbf{u}_0$ is similar to [4, Theorem 1.4]. Both approximations make critical use of L^3_{uloc}.

The fixed point argument for local existence of strong solutions of Lin-Lin-Wang [43] is not dimension dependent. Thus modifying the solution space X of Lin-Lin-Wang in [43, Theorem 3.1] one can consider for $K > 0$, $0 < \alpha < 1$ the space

$$\begin{aligned} X_T = \Big\{ &(\mathbf{u}, \mathbf{d}) : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3 \times \mathbb{R}^3 : \nabla \cdot \mathbf{u} = 0, \\ &\nabla^2 f, \partial_t f \in C_b(\mathbb{R}^3 \times [0, T]) \cap C^\alpha(\mathbb{R}^3 \times [0, T]), \\ &(\mathbf{u}, \mathbf{d})|_{t=0} = (\mathbf{u}^k_0, \mathbf{d}^k_0),\ \|(\mathbf{u} - \mathbf{u}^k_0, \mathbf{d} - \mathbf{d}^k_0)\|_{C^{2,1}_\alpha(\mathbb{R}^3\times[0,1])} \le K \Big\}, \end{aligned}$$

Such considerations allow us to find solutions $(\mathbf{u}^k, P^k, \mathbf{d}^k)$ of (5.1) in $\mathbb{R}^3 \times [0, T_k]$ for $0 < T_k < +\infty$, and the initial data $(\mathbf{u}^k_0, \mathbf{d}^k_0)$, such that $(\mathbf{u}^k, \mathbf{d}^k) \in C^\infty(\mathbb{R}^3 \times [0, T_k], \mathbb{R}^3 \times \mathbb{S}^2) \cap C([0, T_k], L^2(\mathbb{R}^3) \times \dot{W}^{1,2}(\mathbb{R}^3))$. These smooth solution will serve as approximate solutions to (5.1) with $(\mathbf{u}_0, \nabla \mathbf{d}_0) \in L^3_{\mathrm{uloc}}(\mathbb{R}^3)$.

Step 2: Uniform lower bounds. The next to step in the proof of Theorem 5.2 is to find uniform lower bounds on the intervals of existence for the approximating smooth solutions. That is we need to verify the claim:

Claim 5.9. *There exists* $\tau_0 > 0$ *such that if* T_k *is the maximal time interval for the smooth solutions* $(\mathbf{u}^k, \mathbf{d}^k)$ *obtained by approximation process described in* **Step 1** *(that is, from fixed-point iteration), then* $T_k \ge \tau_0$, *and one has that*

$$\sup_{0\le t\le \tau_0} |||(\mathbf{u}^k(t), \nabla \mathbf{d}^k(t)|||^3_{L^{1/2}(\mathbb{R}^3)} \le 2C_0^3 \epsilon_0^3. \tag{5.13}$$

To verify Claim 5.9, we employ a local L^3 energy inequality, specifically:

Lemma 5.10. *There exists $C > 0$ such that for $0 < T \le \infty$, if $(\mathbf{u}, \mathbf{d}) \in C^\infty(\mathbb{R}^3 \times [0,T), \mathbb{R}^3 \times \mathbb{S}^2) \cap C([0,T), L^2(\mathbb{R}^3) \times \dot{W}^{1,2}(\mathbb{R}^3))$ is a smooth solution of the system (5.1), then*

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3}(|\mathbf{u}|^3 + |\nabla\mathbf{d}|^3)\phi^2 + \int_{\mathbb{R}^3}\Big(|\nabla(|\mathbf{u}|^{3/2}\phi)|^2 + |\nabla(|\nabla\mathbf{d}|^{3/2}\phi)|^2\Big) \\ &\le C\int_{\mathbb{R}^3}(|\mathbf{u}|^3 + |\nabla\mathbf{d}|^3)|\nabla\phi|^2 + CR^{-2}\sup_{y\in\mathbb{R}^3}\Big(\int_{B_R(y)}|\mathbf{u}|^3 + |\nabla\mathbf{d}|^3\Big)^{5/3} \\ &\quad + C\Big(\int_{spt\phi}|\mathbf{u}|^3 + |\nabla\mathbf{d}|^3\Big)^{2/3}\int_{\mathbb{R}^3}\Big(|\nabla(|\mathbf{u}|^{3/2}\phi)|^2 + |\nabla(|\nabla\mathbf{d}|^{3/2}\phi)|^2\Big), \end{aligned} \tag{5.14}$$

holds for any $\phi \in C_0^\infty(\mathbb{R}^3)$, with $0 \le \phi \le 1$, the support of ϕ is $spt\phi = B_R(x_0)$ for some $R > 0$ and $x_0 \in \mathbb{R}^3$, and $|\nabla\phi| \le 4R^{-1}$.

Outline of proof. Proof has three basic steps:

1. To obtain the $\nabla\mathbf{d}$ estimates, start by multiplying $(5.1)_3$ by the test function $\phi^2|\nabla\mathbf{d}|\mathbf{d}$ and integrating by parts. Using standard techniques one may estimate the local L^3 energy of $\nabla\mathbf{d}$.
2. Similarly, multiplying $(5.1)_1$ by the test function $\phi^2|\mathbf{u}|\mathbf{u}$ and integrating by parts one may estimate the local L^3 energy of $\mathbf{u}$.
3. Estimate the pressure P. This step requires significant work which is recorded in Lemma 5.11.

□

Lemma 5.11. *Under the same assumptions as in Lemma 5.10, assume that $\phi \in C_0^\infty(\mathbb{R}^3)$ satisfies $0 \le \phi \le 1$, spt $\phi = B_R(x_0)$ for some $x_0 \in \mathbb{R}^3$, and $|\nabla\phi| \le 2R^{-1}$. Then there exists $C > 0$ such that for any $t \in (0,T)$ there is $c(t) \in \mathbb{R}$ so that the following estimate holds*

$$\begin{aligned} \Big(\int_{\mathbb{R}^3}|P(t) - c(t)|^3\phi^3\Big)^{1/3} &\le C\Big(\int_{spt\phi}|\mathbf{u}(t)|^3 + |\nabla\mathbf{d}(t)|^3\Big)^{1/6} \\ &\quad \Big(\int_{\mathbb{R}^3}(|\nabla(|\mathbf{u}(t)|^{3/2}\phi)|^2 + |\nabla(|\nabla\mathbf{d}(t)|^{3/2}\phi)|^2\Big)^{1/2} \\ &\quad + CR^{-1}\sup_{y\in\mathbb{R}^3}\Big(\int_{B_R(y)}|\mathbf{u}(t)|^3 + |\nabla\mathbf{d}(t)|^3\Big)^{2/3}. \end{aligned} \tag{5.15}$$

Outline of proof. The proof of the pressure estimate relies on writing P in terms of Riesz transforms and making a commutator estimate. That is, one has

$$-\Delta P = \nabla^2_{jk}(g^{jk}), \quad g^{jk} := u^j u^k + \nabla_j\mathbf{d}\cdot\nabla_k\mathbf{d},$$

and so

$$P = -\mathbf{R}_j\mathbf{R}_k(g^{jk})$$

where $\mathbf{R}_j$ is the j-th Riesz transform on $\mathbb{R}^3$.

One proceeds by writing

$$\begin{aligned} (P - c)\phi &= -\mathbf{R}_j\mathbf{R}_k(g^{jk})\phi - c\phi \\ &= -\mathbf{R}_j\mathbf{R}_k(g^{jk}\phi) - [\phi, \mathbf{R}_j\mathbf{R}_k](g^{jk}) - c\phi \end{aligned} \tag{5.16}$$

and estimating each part separately. □

Step 3: Uniform estimation of $(\mathbf{u}_k, \mathbf{d}_k)$. For the approximation sequence $(\mathbf{u}^k, P^k, \mathbf{d}^k)$ one has the pressure Poisson equation

$$-\Delta P^k = \operatorname{div}^2(\mathbf{u} \otimes \mathbf{u} + \nabla \mathbf{d}^k \odot \nabla \mathbf{d}^k) \text{ in } \mathbb{R}^3.$$

Using the generalized energy inequality for suitable weak solutions (see below Theorem 5.13), the uniform lower bound found in Claim 5.9 and the pressure estimate used in the local L^3 energy inequality (Lemma 5.11) one has

$$\sup_{0 \le t \le \tau_0} \sup_{x \in \mathbb{R}^3} \|P^k(t) - c_x^k(t)\|_{L^3(B_1(x))} \le C\epsilon_0, \tag{5.17}$$

where $c_x^k(t) \in \mathbb{R}$ depends on both $x \in \mathbb{R}^3$ and $t \in [0, \tau_0]$. From Claim 5.9 and (5.17), we see that for any $x_0 \in \mathbb{R}^3$, $(\mathbf{u}^k, P^k - c_{x_0}^k, \mathbf{d}^k)$ satisfies the conditions of Theorem 5.14 (below) in $P_{\sqrt{\tau_0}}(x_0, \tau_0) := B_{\sqrt{\tau_0}}(x_0) \times [0, \tau_0]$. Hence by Theorem 5.14 we obtain that $(\mathbf{u}^k, \mathbf{d}^k) \in C^\infty(\mathbb{R}^3 \times (0, \tau_0), \mathbb{R}^3 \times \mathbb{S}^2)$, and

$$\sup_k \|(\mathbf{u}^k, \nabla \mathbf{d}^k)\|_{C^m(\mathbb{R}^3 \times [\delta, \tau_0])} \le C(m, \delta, \epsilon_0) \tag{5.18}$$

holds for any $0 < \delta < \tau_0/2$ and $m \ge 0$.

Here Lemma 5.13 and Theorem 5.14 are concerned with the regularity of suitable weak solutions. Specifically, Lemma 5.13 is generalized energy inequality in the spirit of Caffarelli-Kohn-Nirenberg [6]. Here we adapt notion suitable weak solution to (5.1), a similar definition is given by Lin [37] on the Navier-Stokes equation.

Let $0 < T \le \infty$ and $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain.

Definition 5.12. A triplete of functions $(\mathbf{u}, P, \mathbf{d}) : \Omega \times (0, T) \to \mathbb{R}^3 \times \mathbb{R} \times \mathbb{S}^2$ is called a *suitable weak solution* to the system (5.1) in $\Omega \times (0, T)$ if the following properties hold:

(1) $\mathbf{u} \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\Omega \times (0, T))$, $P \in L^{\frac{3}{2}}(\Omega \times (0, T))$ and $\mathbf{d} \in L_t^2 H_x^2(\Omega \times (0, T))$;
(2) $(\mathbf{u}, P, \mathbf{d})$ satisfies the system (5.1) in the sense of distributions; and
(3) $(\mathbf{u}, P, \mathbf{d})$ satisfies the local energy inequality (5.19).

Lemma 5.13. *Suppose that $(\mathbf{u}, \mathbf{d}) \in C^\infty(\Omega \times (0, T), \mathbb{R}^3 \times \mathbb{R} \times \mathbb{S}^2)$ is a solution of (5.1) in $\Omega \times (0, T)$. Then for any nonnegative $\phi \in C_0^\infty(\Omega \times (0, T))$, it holds that*

$$\begin{aligned} &2 \int_{\Omega \times (0,T)} \left(|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^2\right) \phi \\ &\le \int_{\Omega \times (0,T)} \left(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2\right)(\phi_t + \Delta \phi) + \int_{\Omega \times (0,T)} (|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2 + 2P) \mathbf{u} \cdot \nabla \phi \\ &\quad + 2 \int_{\Omega \times (0,T)} \left(\nabla \mathbf{d} \odot \nabla \mathbf{d} - |\nabla \mathbf{d}|^2 \mathbb{I}_3\right) : \nabla^2 \phi \\ &\quad + 2 \int_{\Omega \times (0,T)} \nabla \mathbf{d} \odot \nabla \mathbf{d} : \mathbf{u} \otimes \nabla \phi. \end{aligned} \tag{5.19}$$

With this framework in place it is verified by Hineman-Wang [22] suitable weak solutions of (5.1) with small renormalized energy are smooth. That is:

Theorem 5.14. *For any $\delta > 0$, there exists $\epsilon_0 > 0$ such that $(\mathbf{u}, P, \mathbf{d}) : \Omega \times (0, T) \to \mathbb{R}^3 \times \mathbb{R} \times \mathbb{S}^2$ is a suitable weak solution to (5.1), and satisfies, for $z_0 = (x_0, t_0) \in$*

$\Omega\times(0,T)$ and $P_{r_0}(z_0)\subset\Omega\times(0,T)$,

$$\Big(r_0^{-2}\int_{P_{r_0}(z_0)}|\mathbf{u}|^3\Big)^{1/3}+\Big(r_0^{-2}\int_{P_{r_0}(z_0)}|P|^{\frac{3}{2}}\Big)^{2/3}+\Big(r_0^{-2}\int_{P_{r_0}(z_0)}|\nabla\mathbf{d}|^3\Big)^{1/3}\le\epsilon_0, \tag{5.20}$$

and

$$\|\nabla\mathbf{d}\|_{L_t^\infty L_x^3(P_{r_0}(z_0))}<\frac{1-\delta}{\mathcal{C}(3)^1}, \tag{5.21}$$

then $(\mathbf{u},\mathbf{d})\in C^\infty(P_{\frac{r_0}{4}}(z_0),\mathbb{R}^3\times\mathbb{S}^2)$, and the following estimate holds:

$$\|(\mathbf{u},\mathbf{d})\|_{C^m(P_{\frac{r_0}{4}}(z_0))}\le C(m,r_0,\epsilon_0),\quad\forall\ m\ge 0. \tag{5.22}$$

Remark 5.15. The proof of Theorem 5.14 employs a decay lemma, Lemma 5.16 and potential estimates between Morrey spaces similar to those developed for harmonic maps by Huang and Wang [25]. For the complete details, please consult the recent article [22].

Lemma 5.16. *For any $\delta>0$, there exist $\epsilon_0>0$ and $\theta_0\in(0,\frac{1}{2})$ such that if $(\mathbf{u},P,\mathbf{d}):\Omega\times(0,T)\to\mathbb{R}^3\times\mathbb{R}\times\mathbb{S}^2$ is a suitable weak solution of* (5.1)*, and satisfies, for $z_0=(x_0,t_0)\in\Omega\times(0,T)$ and $P_{r_0}(z_0)\subset\Omega\times(0,T)$, both* (5.20) *and* (5.21)*, then it holds that*

$$\begin{aligned}
&\Big((\theta_0r_0)^{-2}\int_{P_{\theta_0r_0}(z_0)}|\mathbf{u}|^3\Big)^{1/3}+\Big((\theta_0r_0)^{-2}\int_{P_{\theta_0r_0}(z_0)}|P|^{\frac{3}{2}}\Big)^{2/3}\\
&+\Big((\theta_0r_0)^{-2}\int_{P_{\theta_0r_0}(z_0)}|\nabla\mathbf{d}|^3\Big)^{1/3}\\
&\le\frac{1}{2}\Big[\Big(r_0^{-2}\int_{P_{r_0}(z_0)}|\mathbf{u}|^3\Big)^{1/3}+\Big(r_0^{-2}\int_{P_{r_0}(z_0)}|P|^{\frac{3}{2}}\Big)^{2/3}+\Big(r_0^{-2}\int_{P_{r_0}(z_0)}|\nabla\mathbf{d}|^3\Big)^{1/3}\Big].
\end{aligned} \tag{5.23}$$

Outline of Proof. Lemma 5.16 is proven by contradiction. Specifically, a sequence of suitable weak solutions satisfying the decay inequality (5.23) is constructed and it is shown that its $L_t^3L_x^3$-strong limit does not satisfy the decay inequality (5.23). To achieve this goal, one uses

(1) The generalized energy inequality described in Lemma 5.13 to select an L^2-weakly convergent subsequence of suitable weak solutions. The limit of such a subsequence satisfies

$$\begin{gathered}
\partial_t\mathbf{v}-\Delta\mathbf{v}+\nabla Q=0,\\
\nabla\cdot\mathbf{v}=0,\\
\partial_t\mathbf{e}-\Delta\mathbf{e}=0.
\end{gathered}$$

[1]For the embedding $W^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$, where $\Omega\subset\mathbb{R}^3$, define *the optimal Sobolev embedding constant* $\mathcal{C}(3)$ by

$$\mathcal{C}(3)=\inf\big\{\frac{\|\nabla f\|_{L^2(\Omega)}}{\|f\|_{L^6(\Omega)}}:f\neq 0,f\in C_0^\infty(\mathbb{R}^3)\big\}.$$

It is noted that given Ω one may explicitly calculate $\mathcal{C}(3)$.

and hence through the Sobolev inequality and standard estimates on the linear Stokes and heat equations one finds the limit $(\mathbf{v}, Q, \mathbf{e})$ satisfies

$$\begin{gathered}\theta^{-2}\int_{P_\theta}(|\mathbf{v}|^3+|\nabla\mathbf{e}|^3)\le C\theta^3\int_{P_{3/4}}(|\mathbf{v}|^3+|\nabla\mathbf{e}|^3)\le C\theta^3,\quad \theta\in(0,\frac{1}{2})\\ \theta^{-2}\int_{P_\theta}|Q|^{3/2}\le C\theta\int_{P_{3/4}}|Q|^{3/2}\le C\theta\quad \theta\in(0,\frac{1}{2}).\end{gathered}\tag{5.24}$$

(2) Aubin-Lions Lemma [53]:

Lemma 5.17. *Let $X_0 \subset X \subset X_1$ be Banach spaces such that X_0 is compactly embedded in X, X is continuously embedded in X_1, and X_0, X_1 are reflexive. Then for $1 < \alpha_0, \alpha_1 < \infty$,*

$\big\{u \in L^{\alpha_0}(0,T;X_0) : \partial_t u \in L^{\alpha_1}(0,T;X_1)\big\}$ *is compactly embedded in* $L^{\alpha_0}(0,T;X)$.

allows the weak convergence of $v_i \to v$ and $\nabla e_i \to e_i$ to be upgraded to strong convergence in $L^2_tL^2_x$ and after interpolation strong convergence in $L^3_tL^3_x$. To apply the Aubin-Lions lemma one must employ the appropriate $W^{k,p}_\alpha$ estimates for auxiliary Stokes and Heat equations.

(3) Separate pressure estimate:

Lemma 5.18. *Suppose that $(\mathbf{u}, P, \mathbf{d})$ is a suitable weak solution of (5.1) on P_1. Then for any $0 < r \le 1$ and $\tau \in (0, r/2)$, it holds that*

$$\frac{1}{\tau^2}\int_{P_\tau}|P|^{3/2}\le C\Big[\big(\frac{r}{\tau}\big)^2\frac{1}{r^2}\int_{P_r}(|\mathbf{u}-\mathbf{u}_r(t)|^3+|\nabla\mathbf{d}|^3)+\big(\frac{\tau}{r}\big)\frac{1}{r^2}\int_{P_r}|P|^{3/2}\Big],\tag{5.25}$$

where $\mathbf{u}_r(t) = \frac{1}{|B_r|}\int_{B_r}\mathbf{u}(x,t)$ for $-r^2 \le t \le 0$. In particular, it holds that

$$\begin{aligned}\frac{1}{\tau^2}\int_{P_\tau}|P|^{3/2}&\le C\big(\frac{r}{\tau}\big)^2\Big(\sup_{-r^2\le t\le 0}\frac{1}{r}\int_{B_r}|\mathbf{u}|^2\Big)^{3/4}\Big(\frac{1}{r}\int_{P_r}|\nabla u|^2\Big)^{3/4}\\ &\quad+C\Big[\big(\frac{r}{\tau}\big)^2\frac{1}{r^2}\int_{P_r}|\nabla\mathbf{d}|^3+\big(\frac{\tau}{r}\big)\frac{1}{r^2}\int_{P_r}|P|^{3/2}\Big].\end{aligned}\tag{5.26}$$

□

Step 4: Passage to the limit. Based on the estimates of $(\mathbf{u}^k, \mathbf{d}^k)$, we may assume, after taking subsequences, that $(\mathbf{u}, \mathbf{d}) \in \cap_{0<\delta<\tau_0} C^\infty_b(\mathbb{R}^3 \times [\delta, \tau_0], \mathbb{R}^3 \times \mathbb{S}^2)$, with $(\mathbf{u}, \nabla\mathbf{d}) \in L^\infty([0, \tau_0], L^3_{\rm uloc}(\mathbb{R}^3))$, such that

$$\begin{gathered}(\mathbf{u}^k,\nabla\mathbf{d}^k)\to(\mathbf{u},\nabla\mathbf{d})\quad\text{weakly in } L^3(\mathbb{R}^3\times[0,\tau_0]),\\ (\mathbf{u}^k,\mathbf{d}^k)\to(\mathbf{u},\mathbf{d})\quad\text{in } C^m(B_R\times[\delta,\tau_0]),\ \forall\, m\ge 0, R>0,\delta<\tau_0.\end{gathered}$$

Sending $k \to \infty$ one may find that

$$\sup_{0\le t\le\tau_0}|||(\mathbf{u},\nabla\mathbf{d})|||_{L^3_1(\mathbb{R}^3)}\le C\epsilon_0.$$

We can check from (5.1) and that for any $R > 0$,

$$\|(\partial_t\mathbf{u}^k,\partial_t\mathbf{d}^k)\|_{L^{3/2}([0,\tau_0],W^{-1,\frac{3}{2}}(B_R))}\le C(R)<+\infty.$$

This implies that

$$(\mathbf{u}(t),\nabla\mathbf{d}(t))\to(\mathbf{u}_0,\nabla\mathbf{d}_0)\quad\text{strongly in } L^3_{\rm loc}(\mathbb{R}^3)\text{ as } t\downarrow 0.\tag{5.27}$$

In particular, we have that $(\mathbf{u}_0, \nabla\mathbf{d}_0) \in C^0_*([0, \tau_0], L^3_{\rm uloc}(\mathbb{R}^3))$.

Step 5: Characterization of the maximal time interval T_0. Let $T_0 > \tau_0$ be the maximal time interval for the solution $(\mathbf{u}, \mathbf{d})$ constructed in step 4. Suppose that $T_0 < +\infty$ and (5.7) were false. Then there exists $r_0 > 0$ so that

$$\limsup_{t\uparrow T_0} |||(\mathbf{u}(t), \nabla\mathbf{d}(t))|||_{L^3_{r_0}(\mathbb{R}^3)} \le \epsilon_0.$$

In particular, there exists $r_1 \in (0, r_0]$ such that

$$\sup_{T_0 - r_1^2 \le t \le T_0} |||(\mathbf{u}(t), \nabla\mathbf{d}(t))|||_{L^3_{r_1}(\mathbb{R}^3)} \le \epsilon_0.$$

Hence by Theorem 5.14, $(\mathbf{u}, \mathbf{d}) \in C_b^\infty(\mathbb{R}^3 \times [0, T_0]) \cap L^\infty([0, T_0], L^3_{\rm uloc}(\mathbb{R}^3))$. This contradicts the maximality of T_0. Hence (5.7) holds.

Step 6: Uniqueness. Let $(\mathbf{u}_1, \mathbf{d}_1), (\mathbf{u}_2, \mathbf{d}_2) : \mathbb{R}^3 \times [0, T_0] \to \mathbb{R}^3 \times \mathbb{S}^2$ be two solutions of (5.1), under the same initial condition $(\mathbf{u}_0, \mathbf{d}_0)$, that satisfy the properties of Theorem 5.2. We first show $(\mathbf{u}_1, \mathbf{d}_1) \equiv (\mathbf{u}_2, \mathbf{d}_2)$ in $\mathbb{R}^3 \times [0, \tau_0]$. This can be done by the argument of [56] pages 15-16 and is recorded in [22]. We repeat it here for completeness.

Set $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $\mathbf{d} = \mathbf{d}_1 - \mathbf{d}_2$. Then $(\mathbf{u}, \mathbf{d})$ satisfies

$$\begin{gathered}
\partial_t \mathbf{u} - \Delta\mathbf{u} = -\mathbb{P}\nabla\cdot[\mathbf{u}_1 \otimes \mathbf{u}_1 - \mathbf{u}_2 \otimes \mathbf{u}_2 + \nabla\mathbf{d}_1 \odot \nabla\mathbf{d}_1 - \nabla\mathbf{d}_2 \odot \nabla\mathbf{d}_2] \\
\partial_t \mathbf{d} - \Delta\mathbf{d} = -(\mathbf{u}_1 \cdot \nabla\mathbf{d}_1 - \mathbf{u}_2 \cdot \nabla\mathbf{d}_2) + |\nabla\mathbf{d}_1|^2\mathbf{d}_1 - |\nabla\mathbf{d}_2|^2\mathbf{d}_2 \\
(\mathbf{u}, \mathbf{d})|_{t=0} = (0, 0).
\end{gathered}$$

By the Duhamel formula, we have

$$\begin{gathered}
\mathbf{u}(t) = -\mathbb{V}[\mathbf{u}_1 \otimes \mathbf{u}_1 - \mathbf{u}_2 \otimes \mathbf{u}_2 + \nabla\mathbf{d}_1 \odot \nabla\mathbf{d}_1 - \nabla\mathbf{d}_2 \odot \nabla\mathbf{d}_2] \\
\mathbf{d}(t) = -\mathbb{S}[(\mathbf{u}_1 \cdot \nabla\mathbf{d}_1 - \mathbf{u}_2 \cdot \nabla\mathbf{d}_2) - (|\nabla\mathbf{d}_1|^2\mathbf{d}_1 - |\nabla\mathbf{d}_2|^2\mathbf{d}_2)],
\end{gathered}$$

where

$$\begin{gathered}
\mathbb{S}f(t) = \int_0^t e^{-(t-s)\Delta} f(s)\, ds, \\
\mathbb{V}f(t) = \int_0^t e^{-(t-s)\Delta} \mathbb{P}\nabla\cdot f(s)\, ds, \quad \forall f : \mathbb{R}^3 \times [0, +\infty) \to \mathbb{R}^3.
\end{gathered}$$

Recall the three function spaces used in [55]. Let $\mathbf{X}_{\tau_0}$ denote the space of functions f on $\mathbb{R}^3 \times [0, \tau_0]$ such that

$$|||f|||_{\mathbf{X}_{\tau_0}} := \sup_{0<t\le\tau_0} \|f(t)\|_{L^\infty(\mathbb{R}^3)} + \|f\|_{X_{\tau_0}} < +\infty,$$

where

$$\|f\|_{\mathbf{X}_{\tau_0}} := \sup_{0<t\le\tau_0} \sqrt{t}\|\nabla f(t)\|_{L^\infty(\mathbb{R}^3)} + \sup_{x\in\mathbb{R}^3, 0<r\le\sqrt{\tau_0}} (r^{-3}\int_{P_r(x,r^2)} |\nabla f|^2)^{1/2},$$

$\mathbf{Y}_{\tau_0}$ denote the space of functions g on $\mathbb{R}^3 \times [0, \tau_0]$ such that

$$||g||_{\mathbf{Y}_{\tau_0}} := \sup_{0<t\le\tau_0} t\|g(t)\|_{L^\infty(\mathbb{R}^3)} + \sup_{x\in\mathbb{R}^3, 0<r\le\sqrt{\tau_0}} r^{-3}\int_{P_r(x,r^2)} |g| < +\infty,$$

and $\mathbf{Z}_{\tau_0}$ the space of functions h on $\mathbb{R}^3 \times [0, \tau_0]$ such that

$$\|h\|_{\mathbf{Z}_{\tau_0}} := \sup_{0<t\le\tau_0} \sqrt{t}\|h(t)\|_{L^\infty(\mathbb{R}^3)} + \sup_{x\in\mathbb{R}^3, 0<r\le\sqrt{\tau_0}} (r^{-3}\int_{P_r(x,r^2)} |h|^2)^{1/2} < +\infty.$$

Since $(\mathbf{u}_i, \mathbf{d}_i) \in L^\infty([0,\tau_0], L^2(\mathbb{R}^3) \times \dot{W}^{1,2}(\mathbb{R}^3))$ satisfies (5.6) for $i = 1, 2$, Theorem 5.14 and the Hölder inequality imply that $\mathbf{u}_i \in \mathbf{Z}_{\tau_0}, \mathbf{d}_i \in \mathbf{X}_{\tau_0}$ for $i = 1, 2$, and

$$\sum_{i=1}^{2} (\|\mathbf{u}_i\|_{\mathbf{Z}_{\tau_0}} + \|\mathbf{d}_i\|_{\mathbf{X}_{\tau_0}}) \le C\epsilon_0.$$

It follows from Lemma 3.1 and Lemma 4.1 of [55] that

$$\begin{aligned}
&\|\mathbf{u}\|_{\mathbf{Z}_{\tau_0}} + |||\mathbf{d}|||_{\mathbf{X}_{\tau_0}} \\
&\lesssim \Big\| (|\mathbf{u}_1| + |\mathbf{u}_2|)|\mathbf{u}| + (|\nabla \mathbf{d}_1| + |\nabla \mathbf{d}_2|)|\nabla \mathbf{d}| \Big\|_{\mathbf{Y}_{\tau_0}} \\
&\quad + \Big\| |\mathbf{u}||\nabla \mathbf{d}_2| + |\mathbf{u}_1||\nabla \mathbf{d}| + (|\nabla \mathbf{d}_1| + |\nabla \mathbf{d}_2|)|\nabla \mathbf{d}| + |\nabla \mathbf{d}_2|^2 |\mathbf{d}| \Big\|_{\mathbf{Y}_{\tau_0}} \\
&\lesssim \Big[\sum_{i=1}^{2} (\|\mathbf{d}_i\|_{\mathbf{X}_{\tau_0}} + \|\mathbf{u}_i\|_{\mathbf{Z}_{\tau_0}}) \Big] \|\mathbf{u}\|_{\mathbf{Z}_{\tau_0}} + \Big[\sum_{i=1}^{2} (\|\mathbf{u}_i\|_{\mathbf{Z}_{\tau_0}} + \|\mathbf{d}_i\|_{\mathbf{X}_{\tau_0}}) \Big] |||\mathbf{d}|||_{\mathbf{X}_{\tau_0}} \\
&\lesssim \epsilon_0 [\|\mathbf{u}\|_{\mathbf{Z}_{\tau_0}} + |||\mathbf{d}|||_{\mathbf{X}_{\tau_0}}].
\end{aligned}$$

This clearly implies that $(\mathbf{u}_1, \mathbf{d}_1) \equiv (\mathbf{u}_2, \mathbf{d}_2)$ in $\mathbb{R}^3 \times [0, \tau_0]$. Since $(\mathbf{u}_1, \mathbf{d}_1)$ and $(\mathbf{u}_2, \mathbf{d}_2)$ are classical solutions of (5.1) in $\mathbb{R}^3 \times [\tau_0, T_0)$, and $(\mathbf{u}_1, \mathbf{d}_1) = (\mathbf{u}_2, \mathbf{d}_2)$ at $t = \tau_0$, it is standard that $(\mathbf{u}_1, \mathbf{d}_1) \equiv (\mathbf{u}_2, \mathbf{d}_2)$ in $\mathbb{R}^3 \times [\tau_0, T_0)$. The proof is complete.

5.1. **Open problems.** Again, as this analysis is preliminary, there are many open problems surrounding the hydrodynamic flow of nematic liquid crystals in $\mathbb{R}^3$. It seems that the ultimate goal of future investigations should be to re-introduce terms removed from (2.14) to arrive at (5.1). There do remain a number of open problems in the analysis of the system (5.1), we mention a particular one of interest:

(1) **Partial regularity of suitable weak solutions in three dimensions.** In the proof of Theorem 5.14 it was required to prove a quantitative decay lemma, Lemma 5.16. To control certain terms one assumed (5.21) (for further details, please consult [22]). It is an obvious question to ask if the assumption (5.21) can be eliminated. If one can overcome the difficulties associated with the elimination of this condition and, following [6], proves a theorem concerning the rate at which singularities develop for $|\nabla \mathbf{u}|$ and $|\nabla^2 \mathbf{d}|$ one would have the following *partial regularity* theorem.

Conjecture 5.19 (Partial Regularity)**.** *For any suitable weak solution of the system* (5.1) *on an open set in space-time, the set*

$$\mathcal{S} = \text{“the singular set”} = \{(x,t) : \|(\mathbf{u}, \nabla \mathbf{d})\|_{L^\infty_{\rm loc}(P_r(x,t))} = +\infty\}$$

has 1*-dimensional parabolic Hausdorff measure 0; that is,*

$$\mathcal{P}^1(\mathcal{S}) = 0 \text{ where } \mathcal{P}^k(X) := \lim_{\delta \to 0^+} \inf \Big\{ \sum_{i=1}^{\infty} r_i^k : X \subset \bigcup_i P_{r_i}, r_i < \delta \Big\}.$$

Such a result would be analogous to the celebrated work of Caffarelli, Kohn, and Nirenberg in [6] and would extend the work of Lin-Lin-Wang [43] and Hardt-Kinderleher-Lin [19].

Conclusions. Liquid crystals, and in general materials with fine structure will require new tools in mathematics, modeling, and computation. This survey has primarily focused on the mathematical difficulties and open problems surrounding the Ericksen-Leslie equations for the hydrodynamic flow of nematic liquid crystals. We have made the case that there many open problems in the analysis and numerical of these equations that require new mathematics.

It is worth mentioning in closing that there are alternative models of liquid crystals and their flow. For example, the Q-tensor theory of De Gennes [17], [49] and the micropolar theory of Eringen [14]. The analysis and numerical analysis of these models in also preliminary and offers additional directions for research.

References

[1] F. Alouges. A new algorithm for computing liquid crystal stable configurations: the harmonic mapping case. *SIAM journal on numerical analysis*, 34(5):1708-1726, 1997.

[2] F. Alouges and J. M. Ghidaglia. Minimizing Oseen–Frank energy for nematic liquid crystals: algorithms and numerical results. In *Annales de l'IHP Physique théorique*, volume 66, page 411-447, 1997.

[3] S. Bartels. Stability and convergence of finite-element approximation schemes for harmonic maps. *SIAM journal on numerical analysis*, 43(1):220–238, 2005.

[4] A. Basson. Uniformly locally square integrable solutions for 2D Navier-Stokes equations. *Preprint*, 2006.

[5] J. T. Beale, T. Kato, and A. Majda. Remarks on the breakdown of smooth solutions for the 3-d Euler equations. *Communications in Mathematical Physics*, 94(1):61–66, 1984.

[6] L. Caffarelli, R. Kohn, and L. Nirenberg. Partial regularity of suitable weak solutions of the navier-stokes equations. *Communications on Pure and Applied Mathematics*, 35(6):771–831, 1982.

[7] Cecilia Cavaterra and Elisabetta Rocca. On a 3D isothermal model for nematic liquid crystals accounting for stretching terms. *arXiv:1107.3947*, July 2011.

[8] Y. Chen and M. Struwe. Existence and partial regularity results for the heat flow for harmonic maps. *Mathematische Zeitschrift*, 201(1):83–103, 1989.

[9] Yunmei Chen. The weak solutions to the evolution problems of harmonic maps. *Mathematische Zeitschrift*, 201(1):69–74, 1989.

[10] Alexandre J. Chorin and Jerrold E. Marsden. *A Mathematical Introduction to Fluid Mechanics (Texts in Applied Mathematics)*. Springer, 3rd edition, May 1993.

[11] Robert Cohen and Michael Taylor. Weak stability of the map x/|x| for liquid crystal functionals. *Communications in Partial Differential Equations*, 15(5):675–692, January 1990.

[12] Daniel Coutand and Steve Shkoller. Well-posedness of the full EricksenLeslie model of nematic liquid crystals. *Comptes Rendus de l'Acadmie des Sciences - Series I - Mathematics*, 333(10):919–924, November 2001.

[13] J. L. Ericksen. Conservation laws for liquid crystals. *Journal of rheology*, 5:23, 1961.

[14] A. C. Eringen. A unified continuum theory of liquid crystals. *ARI-An International Journal for Physical and Engineering Sciences*, 50(2):73–84, 1997.

[15] L Escauriaza, G A Seregin, and Vladimir Sverak. $L_{3,\infty}$-solutions of the Navier-Stokes equations and backward uniqueness. *Russian Mathematical Surveys*, 58(2):211–250, April 2003.

[16] H. Fujita and T. Kato. On the Navier-Stokes initial value problem. i. *Archive for rational mechanics and analysis*, 16(4):269–315, 1964.

[17] P. G. de Gennes and J. Prost. *The Physics of Liquid Crystals*. Clarendon Press, April 1995.

[18] P. Germain. Multipliers, paramultipliers, and weak-strong uniqueness for the Navier-Stokes equations. *Journal of Differential Equations*, 226(2):373–428, 2006.

[19] Robert Hardt, David Kinderlehrer, and Fang-Hua Lin. Existence and partial regularity of static liquid crystal configurations. *Communications in Mathematical Physics (1965-1997)*, 105(4):547–570, 1986.

[20] Robert Hardt and Fang-Hua Lin. Stability of singularities of minimizing harmonic maps. *Journal of Differential Geometry*, 29(1):113–123, 1989.

[21] Jay Hineman. The hydrdynamic flow of nematic liquid crystals in R^3. *Theses and Dissertations Mathematics*, January 2012.
[22] Jay Hineman and Changyou Wang. Well-posedness of nematic liquid crystal flow in $L^3_{\mathrm{uloc}}(\mathbb{R}^3)$ *arXiv:1208.5965*, August 2012.
[23] Min-Chun Hong. Global existence of solutions of the simplified Ericksen-Leslie system in dimension two. *Calculus of Variations and Partial Differential Equations*, 40(1):15–36, 2011.
[24] Min-Chun Hong and Zhouping Xin. Global existence of solutions of the liquid crystal flow for the oseen-frank model. *arXiv:1010.4083*, October 2010.
[25] T. Huang and C. Y. Wang. Notes on the regularity of harmonic map systems. In *Proc. Amer. Math. Soc*, volume 138, page 2015–2023, 2010.
[26] Tao Huang and Changyou Wang. Blow up criterion for nematic liquid crystal flows. *Communications in Partial Differential Equations*, 37(5):875–884, 2012.
[27] Tao Huang and Changyou Wang. On uniqueness of heat flow of harmonic maps. *arXiv: 1208.1470*, August 2012.
[28] Tao Huang and Wang, Changyou. Regularity and uniqueness for a class of solutions to the hydrodynamic flow of nematic liquid crystals. *Preprint*, 2012.
[29] F. Hélein. Minima de la fonctionelle energie libre des cristaux liquides. *CR Acad. Sci. Paris*, 305:565–568, 1987.
[30] T. Kato. Strong L^p-solutions of the Navier-Stokes equation R^m, with applications to weak solutions. *Mathematische Zeitschrift*, 187(4):471–480, 1984.
[31] D. Kinderlehrer, B. Ou, D. Kinderlehrer, and B. Ou. Second variation of liquid crystal energy at $x/|x|\dagger$. *Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences*, 437(1900):475–487, 1992.
[32] H. Koch and D. Tataru. Well-posedness for the NavierStokes equations. *Advances in Mathematics*, 157(1):22–35, 2001.
[33] P. G. Lemarie-Rieusset. *Recent developments in the Navier-Stokes problem.* CRC Press, April 2002.
[34] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta mathematica*, 63(1):193–248, 1934.
[35] F. M. Leslie. Some constitutive equations for liquid crystals. *Archive for Rational Mechanics and Analysis*, 28(4):265–283, 1968.
[36] Xiaoli Li and Dehua Wang. Global solution to the incompressible flow of liquid crystals. *arXiv:1108.5477*, August 2011.
[37] F. Lin. A new proof of the Caffarelli–Kohn–Nirenberg theorem. *Communications on Pure and Applied Mathematics*, 51(3):241–257, 1998.
[38] F. H. Lin and C. Liu. Nonparabolic dissipative systems modeling the flow of liquid crystals. *Communications on Pure and Applied Mathematics*, 48(5):501–537, 1995.
[39] F. H. Lin and C. Liu. Partial regularity of the dynamic system modeling the flow of liquid crystals. *DYNAMICAL SYSTEMS*, 2(1):1–22, 1996.
[40] Fang-Hua Lin. Nonlinear theory of defects in nematic liquid crystals; phase transition and flow phenomena. *Communications on Pure and Applied Mathematics*, 42(6):789–814, 1989.
[41] Fang-Hua Lin and Chun Liu. Existence of solutions for the Ericksen-Leslie system. *Archive for Rational Mechanics and Analysis*, 154(2):135–156, 2000.
[42] Fanghua Lin. Some analytical issues for elastic complex fluids. *Communications on Pure and Applied Mathematics*, 65(7):893919, 2012.
[43] Fanghua Lin, Junyu Lin, and Changyou Wang. Liquid crystal flows in two dimensions. *Archive for Rational Mechanics and Analysis*, 197(1):297–336, November 2009.
[44] Fanghua Lin and Chun Liu. Static and dynamic theories of liquid crystals. *J. Partial Differential Equations*, 14(4):289–330, 2001.
[45] Fanghua Lin and Changyou Wang. On the uniqueness of heat flow of harmonic maps and hydrodynamic flow of nematic liquid crystals. *Chinese Annals of Mathematics*, 31(6):921–938, November 2010.
[46] Junyu Lin and Shijin Ding. On the well-posedness for the heat flow of harmonic maps and the hydrodynamic flow of nematic liquid crystals in critical spaces. *Mathematical Methods in the Applied Sciences*, 35(2):158–173, January 2012.
[47] S. Y. Lin and M. Luskin. Relaxation methods for liquid crystal problems. *SIAM Journal on Numerical Analysis*, 26(6):13101324, 1989.

[48] Pierre-Louis Lions. *Mathematical Topics in Fluid Mechanics: Volume 1: Incompressible Models*. Oxford University Press, USA, August 1996.
[49] Apala Majumdar and Arghir Zarnescu. Landau–de Gennes theory of nematic liquid crystals: the 0seen–Frank limit and beyond. *Archive for Rational Mechanics and Analysis*, 196(1):227–280, 2010.
[50] J. Serrin. The initial value problem for the Navier-Stokes equations. *Nonlinear problems*, page 69–98, 1963.
[51] Iain W. Stewart. *The Static and Dynamic Continuum Theory of Liquid Crystals: A Mathematical Introduction*. CRC Press, June 2004.
[52] Huan Sun and Chun Liu. On energetic variational approaches in modeling the nematic liquid crystal flows. *Discrete and Continuous Dynamical Systems*, 23(1/2):455–475, September 2008.
[53] Roger Temam. *Navier-Stokes Equations: Theory and Numerical Analysis*. American Mathematical Soc., 2001.
[54] E. G. Virga. *Variational Theories for Liquid Crystals*. CRC Press, May 1995.
[55] C. Wang. Well-posedness for the heat flow of harmonic maps and the liquid crystal flow with rough initial data. *Archive for Rational Mechanics and Analysis*, 200(1):1–19, 2011.
[56] Changyou Wang. Well-posedness for the heat flow of harmonic maps and the liquid crystals flow with rough initial data. *arXiv:1001.2296*, January 2010.
[57] H. Wen and S. Ding. Solutions of incompressible hydrodynamic flow of liquid crystals. *Nonlinear Analysis: Real World Applications*, 12(3):1510–1531, 2011.
[58] Hao Wu, Xiang Xu, and Chun Liu. Asymptotic behavior for a nematic liquid crystal model with different kinematic transport properties. *arXiv:0901.1751*, January 2009.
[59] Hao Wu, Xiang Xu, and Chun Liu. On the general Ericksen-Leslie system: Parodi's relation, well-posedness and stability. *arXiv:1105.2180*, May 2011.
[60] X. Xu and Z. Zhang. Global regularity and uniqueness of weak solution for the 2-d liquid crystal flows. *Journal of Differential Equations*, 252(2):1169–1181, 2012.

Jay Lawrence Hineman
Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA
E-mail address: jay.hineman@uky.edu