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BOGDANOV-TAKENS SINGULARITY OF A NEURAL NETWORK MODEL WITH DELAY

XIAOQIN P. WU

Abstract. In this article, we study Bogdanov-Takens (BT) singularity of a tree-neuron model with time delay. By using the frameworks of Campbell-Yuan [2] and Faria-Magalhães [4, 5], the normal form on the center manifold is derived for this singularity and hence the corresponding bifurcation diagrams such as Hopf, double limit cycle, and triple limit cycle bifurcations are obtained. Examples are given to verify some theoretical results.

1. Introduction

The objective of this manuscript is to study codimension-2 (Bogdanov-Takens (BT)) bifurcation of the tree-neuron model with delay

$$\begin{aligned} \frac{dv_1(t)}{dt} &= -v_1(t) + f_1(v_3(t) - bv_3(t-\tau)), \\ \frac{dv_2(t)}{dt} &= -v_2(t) + f_2(v_1(t) - bv_1(t-\tau)), \\ \frac{dv_3(t)}{dt} &= -v_3(t) + f_3(v_2(t) - bv_2(t-\tau)). \end{aligned} \tag{1.1}$$

Here f_i is a C^4 functions with $f_i(0) = 0$ $(i = 1, 2, 3)$, $a_i = f_i'(0) > 0$ corresponds to the range of the continuous variable v_i, $b > 0$ is the measure of the inhibitory influence of the past history, and $\tau > 0$ is the time delay due to the time for other neurons to respond. This model is a little bit different from the ones studied in [1, 3, 6, 7, 9] in which our functions $f_i(x)$ $(i = 1, 2, 3)$ can be different.

Neural networks or neural nets have been studied by many researchers since Hopfield [7] constructed a simplified neural network model of a linear circuit consisting of a resistor and a capacitor connected to other neurons via nonlinear sigmoidal activation functions and have been applied to artificial neural network and artificial brain and other fields. In this article, we focus on System (1.1).

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Let $u_i(t) = v_i(t) - bv_i(t-\tau)$ $(i = 1,2,3)$. Then (1.1) can be written as

$$\begin{aligned}\frac{du_1(t)}{dt} &= -u_1(t) + f_1(u_3(t)) - bf_1(u_3(t-\tau)),\\ \frac{du_2(t)}{dt} &= -u_2(t) + f_2(u_1(t)) - bf_2(u_1(t-\tau)),\\ \frac{du_3(t)}{dt} &= -u_3(t) + f_3(u_2(t)) - bf_3(u_2(t-\tau)).\end{aligned} \tag{1.2}$$

Clearly $(0,0,0)$ is an equilibrium point of (1.2) and hence the linearized system at $(0,0,0)$ is

$$\begin{aligned}\frac{du_1(t)}{dt} &= -u_1(t) + a_1u_3(t) - ba_1u_3(t-\tau),\\ \frac{du_2(t)}{dt} &= -u_2(t) + a_2u_1(t) - ba_2u_1(t-\tau),\\ \frac{du_3(t)}{dt} &= -u_3(t) + a_3u_2(t) - ba_3u_2(t-\tau),\end{aligned}$$

whose corresponding characteristic equation is

$$\Delta(\lambda) = (\lambda+1)^3 - a^3(1-be^{-\lambda\tau})^3 = 0 \tag{1.3}$$

where $a^3 = a_1a_2a_3$ and $a_i = f_i'(0)$ $(i = 1,2,3)$.

The dynamical behavior and bifurcation of (1.2) have been studied extensively [1, 3, 6, 9]. In [1, 6, 9], Hopf singularity was studied for $f_i(x) = \tanh(x)$ by using τ as bifurcation parameter. In [3], the authors found critical values of b and τ such that a zero-Hopf singularity occurs.

Note that all the results mentioned above depend on the distribution of roots of the characteristic equation (1.3). If (1.3) has a pair of purely imaginary roots, a Hopf singularity occurs and hence a limit cycle may bifurcate from the equilibrium point. If (1.3) has a simple zero root and a pair of purely imaginary roots, a zero-Hopf singularity occurs. However, under certain conditions, the characteristic equation may have double zero root and this has not been studied in the literature. For a double zero eigenvalue, the corresponding Jordan matrix is either $\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$. Our study shows that only the latter case occurs for (1.2). More specifically, we use (b,τ) as bifurcation parameter to obtain the critical value (b^*,τ^*) such that the characteristic equation has a double zero and then investigate its corresponding dynamical behaviors. Note that we can find the conditions such that the equilibrium point is asymptotically stable. But this is not practical since cyclic behaviors are very common in real world. This leads to study Hopf singularity and in many cases the condition for Hopf singularity is not always satisfied. We show that, for double zero singularity, we still can obtain limit cycles under small perturbations of (b^*,τ^*) and under certain conditions despite of the fact that the condition for Hopf singularity is violated. It turns out that double zero singularity has rich dynamical behaviors. We use the frameworks of Campbell-Yuan [2] and Faria-Magalhães [4, 5] to conduct the center manifold reduction to obtain the normal form for this singularity and hence the corresponding bifurcation diagrams such as Hopf, double limit cycle, and triple limit cycle bifurcations.

The rest of this manuscript is organized as follows. In Section 2, the detailed conditions are given for the linear part of (1.2) at an equilibrium point in the (b,τ)-parameter space to have a triple zero eigenvalue and other eigenvalues with negative real parts. In Section 3, the normal form of double zero singularity for (1.2) is obtained on the center manifold by using the frameworks from [2] and [4, 5]. In Section 4, the normal form in Section 3 is used to obtain bifurcation diagrams of the original (1.2) such as Hopf and homoclinic bifurcations, and two examples are presented to confirm some theoretical results.

2. Distribution of eigenvalues

In the rest of this manuscript, we assume $b = b^* \equiv \frac{a-1}{a}$ and $a > 1$. Clearly if $\tau = \tau^* \equiv \frac{1}{a-1}$, we have $\Delta(0) = \Delta'(0) = 0$ and $\Delta''(0) = \frac{3}{a-1} \neq 0$. Namely $\Delta(\lambda) = 0$ has a zero root of with multiplicity 2 if $\tau = \tau^*$. Clearly (1.3) is equivalent to the equations

$$\lambda - (a - (a-1)e^{-\lambda\tau})e^{2k\pi i/3} = 0, \quad k = 0, 1, 2.$$

For $\omega > 0$, letting $\Delta(i\omega) = 0$, we have

$$1 + i\omega - (a - (a-1)e^{-i\omega\tau}) = 0, \tag{2.1}$$

or

$$1 + i\omega - (a - (a-1)e^{-i\omega\tau})e^{\frac{2\pi}{3}i} = 0, \tag{2.2}$$

or

$$1 + i\omega - (a - (a-1)e^{-i\omega\tau})e^{\frac{4\pi}{3}i} = 0. \tag{2.3}$$

From(2.1), after separating the real part from imaginary part, we have

$$\cos(\omega\tau) = 1, \quad \sin(\omega\tau) = \frac{\omega}{a-1}$$

which give $\omega = 0$. Similarly, from (2.2), we have

$$\cos(\omega\tau) = \frac{1 + 2a - \sqrt{3}\omega}{2(a-1)}, \quad \sin(\omega\tau) = -\frac{\sqrt{3} + \omega}{2(a-1)}.$$

Using $\cos^2(\omega\tau) + \sin^2(\omega\tau) = 1$, we obtain

$$4\omega^2 - a\sqrt{3}\omega + 3a = 0.$$

Clearly if $a < 4$, this equation does not have positive roots. If $a > 4$, it has two different positive roots

$$\omega_\pm = \frac{\sqrt{3}}{2}(a \pm \sqrt{a(a-4)}).$$

In this case, define

$$\tau_j^\pm(a) = \frac{1}{\omega_\pm}[2(j+1)\pi - \arccos\frac{1 + 2a - \sqrt{3}\omega_\pm}{2(a-1)}], \quad j = 0, 1, 2, \dots.$$

If $a = 4$ it has a positive root $\omega = \omega^* \equiv 2\sqrt{3}$ with multiplicity 2. In this case, define

$$\tau_j = \frac{1}{2\sqrt{3}}[2(j+1)\pi - \frac{\pi}{3}], \quad j = 0, 1, 2, \dots.$$

Note that if ωi is a root of (2.2), then $-\omega i$ is a root of (2.3). Now define

$$\gamma = \{(a,\tau) : \tau = \frac{1}{a-1}, a > 1\}, \quad l = \{(a,\tau) : a = 4, \tau > 0\},$$
$$\Gamma_j^\pm = \{(a,\tau) : \tau = \tau_j^\pm(a), a > 4\},$$

and $P_j = (4, \tau_j), j = 0, 1, 2, \dots$. Thus we obtain the following result (see Figure 1 for $j = 0$).

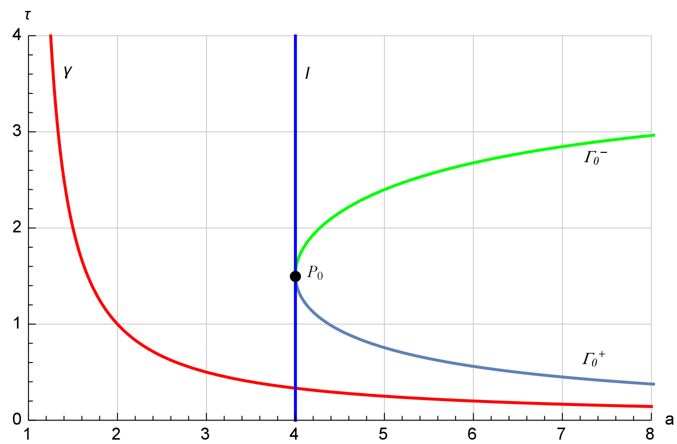

Figure 1. Bifurcation diagrams in (a,τ)-plane for $b = b^*$

Theorem 2.1. *Let $b = \frac{a-1}{a}$ and $a > 1$. In the (a,τ)-plane, we have*

- (i) *if (a,τ) is on the curve γ, the characteristic equation (1.3) has a double zero root and hence a BT (double zero) singularity occurs;*
- (ii) *if (a,τ) is on one of the curves $\Gamma_j^+ = \{(a,\tau) : \tau = \tau_j^+(a)\}$ (or $\Gamma_j^- = \{(a,\tau) : \tau = \tau_j^-(a)\}$) ($j = 0,1,2,\dots$), the characteristic equation (1.3) has a simple zero root and a pair of purely imaginary roots $\pm\omega_+ i$ ($\pm\omega_- i$) and hence a zero-Hopf singularity occurs;*
- (iii) *if (a,τ) is one of the points $(4,\tau_j)$ ($j = 0,1,2,\dots$), the characteristic equation (1.3) has a simple zero root and a pair of purely imaginary roots $2\sqrt{3}$ with multiplicity 2 and hence zero-Hopf 1:1 singularity occurs.*
- (iv) *if (a,τ) is not one of the above, then the characteristic equation (1.3) has a simple zero root and hence a zero (or fold) singularity occurs.*

For the distribution of the rest of eigenvalues for $\tau > 0$, we use the following lemma.

Lemma 2.2 (Ruan and Wei [10]). *Consider the transcendental polynomial*

$$P(\lambda, e^{-\lambda\tau_1}, e^{-\lambda\tau_2}) = p(\lambda) + q_1(\lambda)e^{-\lambda\tau_1} + q_2(\lambda)e^{-\lambda\tau_2},$$

where p, q_1, q_2 are real polynomials such that $\max\{\deg q_1, \deg q_2\} < \deg(p)$ and $\tau_1, \tau_2 \geq 0$. Then as (τ_1,τ_2) varies, the sum of the orders of the zeros of P in the open right half plane can change only if a zero appears on or crosses the imaginary axis.

Note that

$$\Delta(\lambda)_{b=b^*,\tau=0} = \lambda(\lambda^2 + 3\lambda + 3)$$

whose non-zero roots have negative real parts. By using Lemma 2.2, we obtain the following result regarding the rest of eigenvalues.

Lemma 2.3. *If $b = b^*$, all roots of the characteristic equation (1.3) have negative real parts except zero-root and purely imaginary roots.*

We remark that in this manuscript, we only study BT singularity.

3. Computation of the normal form of double zero singularity

In this section, we use the theory of center manifold reduction for general delay differential equations (DDEs) (see the detail in [4, 5]) to compute the normal form of BT singularity. In the rest of this manuscript, we always assume that the assumption (H1) holds. Now we treat (b, τ) as a bifurcation parameter near (b^*, τ^*). By scaling $t \to t/\tau$, (1.2) can be written as

$$\begin{aligned}
\frac{du_1(t)}{dt} &= \tau(-u_1(t) + f_1(u_3(t)) - bf_1(u_3(t-1))),\\
\frac{du_2(t)}{dt} &= \tau(-u_2(t) + f_2(u_1(t)) - bf_2(u_1(t-1))),\\
\frac{du_3(t)}{dt} &= \tau(-u_3(t) + f_3(u_2(t)) - bf_3(u_2(t-1))).
\end{aligned}$$

Let

$$f_i(x) = a_i x + \frac{1}{2} f_i''(0)x^2 + \frac{1}{3!} f'''(0)x^3 + O(x^4).$$

Define $C := C([-1, 0], \mathbb{R}^3)$, $C^* := C([0, 1], \mathbb{R}^{3*})$ and $C^1 = C^1([-1, 0], \mathbb{R}^3)$. Let $\mu_1 = b - b^*$, $\mu_2 = \tau - \tau^*$. Then on C we have

$$\begin{aligned}
\frac{du_1(t)}{dt} &= (\tau^* + \mu_2)\Big[- u_1(0) + a_1u_3(0) - a_1(b^* + \mu_1)u_3(-1) + \frac{1}{2} f_1''(0)u_3^2(0)\\
&\quad - \frac{1}{2}(b^* + \mu_1)f_1''(0)u_3^2(-1) + \frac{1}{6} f_1'''(0)u_3^3(0)\\
&\quad - \frac{1}{6}(b^* + \mu_1)f_1'''(0)u_3^3(-1)\Big] + O(\|\mu\|^2 + \|\mu\|\|y\|^3),\\
\frac{du_2(t)}{dt} &= (\tau^* + \mu_2)\Big[- u_2(0) + a_2u_1(0) - a_2(b^* + \mu_1)u_1(-1) + \frac{1}{2} f_2''(0)u_1^2(0)\\
&\quad - \frac{1}{2}(b^* + \mu_1)f_2''(0)u_1^2(-1) + \frac{1}{6} f_2'''(0)u_1^3(0)\\
&\quad - \frac{1}{6}(b^* + \mu_1)f_2'''(0)u_1^3(-1)\Big] + O(\|\mu\|^2 + \|\mu\|\|y\|^3),\\
\frac{du_3(t)}{dt} &= (\tau^* + \mu_2)\Big[- u_3(0) + a_3u_2(0) - a_3(b^* + \mu_1)u_2(-1) + \frac{1}{2} f_3''(0)u_2^2(0)\\
&\quad - \frac{1}{2}(b^* + \mu_1)f_3''(0)u_2^2(-1) + \frac{1}{6} f_3'''(0)u_2^3(0)\\
&\quad - \frac{1}{6}(b^* + \mu_1)f_3'''(0)u_2^3(-1)] + O(\|\mu\|^2 + \|\mu\|\|y\|^3).
\end{aligned} \tag{3.1}$$

Let

$$\mathbb{A} = \begin{pmatrix} -\frac{1}{a-1} & 0 & \frac{a_1}{a-1}\\ \frac{a_2}{a-1} & -\frac{1}{a-1} & 0\\ 0 & \frac{a_3}{a-1} & -\frac{1}{a-1} \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} 0 & 0 & -\frac{a_1}{a}\\ -\frac{a_2}{a} & 0 & 0\\ 0 & -\frac{a_3}{a} & 0 \end{pmatrix}.$$

Define

$$\Delta(\lambda) = \lambda I - (\mathbb{A} + \mathbb{B}e^{-\lambda}),$$

and the linear operator

$$\mathcal{L}X_t = \mathbb{A}X(t) + \mathbb{B}X(t-1), \quad \text{for } X \in C.$$

From Section 2, we see that $\mathcal{L}$ has a double zero eigenvalue and all the other eigenvalues have negative real parts. It is easy to see that

$$\Delta(0) = -(\mathbb{A} + \mathbb{B}), \quad \Delta'(0) = I + \mathbb{B}.$$

Let $u = (u_1, u_2, u_3)^T \in C$, $\mu = (\mu_1, \mu_2)^T$, and

$$F(u_t, \mu) = (F^1(u_t, \mu), F^2(u_t, \mu), F^3(u_t, \mu))^T,$$

where

$$\begin{aligned} F^1(u_t, \mu) &= -\mu_1 a_1 \tau^* u_3(-1) + \mu_2[-u_1(0) - a_1 b^* u_3(-1)] + \frac{1}{2} f_1''(0) u_3^2(0) \\ &\quad - \frac{1}{2} b^* f_1''(0) u_3^2(-1) + \frac{1}{6} f_1'''(0) u_3^3(0) - \frac{1}{6} b^* f_1'''(0) u_3^3(-1) \\ &\quad + O(\|\mu\|^2 + \|\mu\| \|y\|^3), \\ F^2(u_t, \mu) &= -\mu_1 a_2 \tau^* u_1(-1) + \mu_2[-u_2(0) - a_2 b^* u_1(-1)] + \frac{1}{2} f_2''(0) u_1^2(0) \\ &\quad - \frac{1}{2} (b^* + \mu_1) f_2''(0) u_1^2(-1) + \frac{1}{6} f_2'''(0) u_1^3(0) - \frac{1}{6} b^* f_2'''(0) u_1^3(-1) \\ &\quad + O(\|\mu\|^2 + \|\mu\| \|y\|^3), \\ F^3(u_t, \mu) &= -\mu_1 a_3 \tau^* u_2(-1) + \mu_2[-u_3(0) - a_3 b^* u_2(-1)] + \frac{1}{2} f_3''(0) u_2^2(0) \\ &\quad - \frac{1}{2} b^* f_3''(0) u_2^2(-1) + \frac{1}{6} f_3'''(0) u_2^3(0) - \frac{1}{6} b^* f_3'''(0) u_2^3(-1) \\ &\quad + O(\|\mu\|^2 + \|\mu\| \|y\|^3). \end{aligned}$$

Then (3.1) can be written as

$$\dot{u}(t) = \mathcal{L} u_t + F(u_t, \mu) \tag{3.2}$$

whose corresponding linear part at 0 is

$$\dot{u}(t) = \mathcal{L} u_t. \tag{3.3}$$

From [2], the bilinear form between C and C^* can be expressed as

$$(\psi, \varphi) = \psi(0) \cdot \varphi(0) + \int_{-1}^{0} \psi(\xi + 1) \mathbb{B} \varphi(\xi) d\xi. \tag{3.4}$$

Then $\mathcal{L}$ has a generalized eigenspace P which is invariant under the flow (3.3). Let P^* be the space adjoint with P in C^*. Then C can be decomposed as $C = P \oplus Q$ where $Q = \{\varphi \in C : \langle \psi, \varphi \rangle = 0, \forall \psi \in P^*\}$. Furthermore, we can choose the bases Φ and Ψ for P and P^*, respectively, such that

$$(\Psi, \Phi) = I, \quad \dot{\Phi} = \Phi J, \quad \dot{\Psi} = -J \Psi,$$

where I is the identity matrix and $J = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ the Jordan matrix associated with the double zero eigenvalue with geometric multiplicity 1.

Next, we obtain the explicit expressions of Φ and Ψ. According to Campbell and Yuan [2], the basis Φ for P can be chosen as

$$\Phi = [\varphi_1, \varphi_2] = [v_1, v_2 + \theta v_1]$$

and the basis Ψ for P^* as

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -w_1 s + w_2 \\ w_1 \end{pmatrix}$$

where $v_1, v_2 \in \mathbb{R}^3$ and $w_1, w_2 \in \mathbb{R}^{3*}$ satisfy

$$\Delta(0)v_1 = 0, \Delta'(0)v_1 + \Delta(0)v_2 = 0, \tag{3.5}$$

$$w_1\Delta(0) = 0, w_1\Delta'(0) + w_2\Delta(0) = 0. \tag{3.6}$$

Note that (3.5) is equivalent to

$$(\mathbb{A} + \mathbb{B})v_1 = 0, (\mathbb{A} + \mathbb{B})v_2 = (I + \mathbb{B})v_1,$$

from which we obtain

$$v_1 = \begin{pmatrix} 1 \\ a_2/a \\ a/a_1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ a_2/a \\ a/a_1 \end{pmatrix}$$

Similarly, (3.6) is equivalent to, respectively,

$$w_1(\mathbb{A} + \mathbb{B}) = 0, w_2(\mathbb{A} + \mathbb{B}) = w_1(I + \mathbb{B}).$$

In fact, we have $w_1 = k_1(1, a/a_2, a_1/a)$, $w_2 = k_2(1, a/a_2, a_1/a)$. We can choose $k_1 = 2/3$, $k_2 = -4/9$ such that

$$(\Phi, \Psi) = I.$$

Thus we obtain the bases Φ and Ψ of P and P^* such that $\dot{\Phi} = \Phi J$ and $\dot{\Psi} = -J\Psi$.

Next we compute the corresponding normal form. Let $u = \Phi x + y$ (here $x = (x_1, x_2)^T \in \mathbb{R}^2$ and $y = (y_1, y_2, y_3)^T \in C$); namely

$$\begin{aligned} u_1(\theta) &= x_1 + \theta x_2 + y_1(\theta), \\ u_2(\theta) &= \frac{a_2}{a}x_1 + \frac{a_2(1+\theta)}{a}x_2 + y_2(\theta), \\ u_3(\theta) &= \frac{a}{a_1}x_1 + \frac{a(1+\theta)}{a_1}x_2 + y_3(\theta). \end{aligned}$$

Then, on the center manifold $y = g(x(t), \theta)$, (3.2) becomes

$$\begin{aligned} \dot{x} &= Jx + \Psi(0)F(\Phi x + g(x, \theta), \mu) \\ &= Jx + \frac{1}{2}f_2^1(x, 0, \mu) + \frac{1}{3!}f_3^1(x, 0, \mu) + O(|\mu||x|^2 + |\mu|^2|x| + |x|^4) \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} \frac{1}{2}f_2^1(x, 0, \mu) = &\Big(\frac{4a}{3(a-1)}\mu_1 x_1 - \frac{4}{3}(a-1)\mu_2 x_2, -\frac{2a}{a-1}\mu_1 x_1 + 2(a-1)\mu_2 x_2\Big) \\ &+ \frac{1}{9a_1^2 a_2(a-1)a^4}(a_2 a^5 f_1''(0) + a_1^2 a^4 f_2''(0) \\ &+ a_1^3 a_2^3 f_3''(0))(x_1^2 + 2ax_1x_2 + ax_2^2)\begin{pmatrix} -2 \\ 3 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} &\frac{1}{3!}f_3^1(x, 0, 0) \\ &= \frac{2(a^7 a_2 f_1'''(0) + a^5 a_1^3 f_2'''(0) + a_1^4 a_2^4 f_3'''(0))}{27(a-1)a^5 a_1^3}(x_1^3 + 3ax_1^2 x_2 + 3ax_1 x_2^2 + x_2^3)\begin{pmatrix} -2 \\ 3 \end{pmatrix}. \end{aligned}$$

Using the result in [2] to project on the center manifold up to the second order and after long calculation (we omit the detail), then (3.7) can be transformed as the following normal form,

$$\dot{x} = Jx + \frac{1}{2!}g_2^1(x, 0, \mu) + O(|\mu|^2|x| + |x|^3),$$

or

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \chi_1 x_1 + \chi_2 x_2 + A_{20} x_1^2 + A_{11} x_1 x_2 + O(|\mu|^2|x| + |x|^3), \end{aligned} \tag{3.8}$$

in which χ_j and A_{jk} are given by

$$\begin{gathered} \chi_1 = -\frac{2a}{a-1}\mu_1, \\ \chi_2 = \frac{4a}{3(a-1)}\mu_1 + 2(a-1)\mu_2, \\ A_{20} = \frac{a^5 a_2 f_1''(0) + a^4 a_1^2 f_2''(0) + a_1^3 a_2^3 f_3''(0)}{3a^4 a_1^2 a_2 (a-1)}, \\ A_{11} = \frac{2(3a-2)(a^5 a_2 f_1''(0) + a^4 a_1^2 f_2''(0) + a_1^3 a_2^3 f_3''(0))}{9a^4 a_1^2 a_2 (a-1)}, \end{gathered}$$

if $a^5 a_2 f_1''(0) + a^4 a_1^2 f_2''(0) + a_1^3 a_2^3 f_3''(0) \neq 0$. Since

$$|\frac{\partial \chi}{\partial \mu}| = \det \begin{pmatrix} \frac{\partial \chi_1}{\partial \mu_1} & \frac{\partial \chi_1}{\partial \mu_2} \\ \frac{\partial \chi_2}{\partial \mu_1} & \frac{\partial \chi_2}{\partial \mu_2} \end{pmatrix} = -4a \neq 0,$$

we have that $(\mu_1, \mu_2) \to (\chi_1, \chi_2)$ is regular and hence the transversality condition holds. If $a^5 a_2 f_1''(0) + a^4 a_1^2 f_2''(0) + a_1^3 a_2^3 f_3''(0) = 0$, then (3.7) can be transformed as the following normal form with the third order,

$$\dot{x} = Jx + \frac{1}{2!} g_2^1(x, 0, \mu) + \frac{1}{3!} g_3^1(x, 0, \mu) + O(|\mu|^2|x| + |x|^4),$$

or

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \chi_1 x_1 + \chi_2 x_2 + A_{30} x_1^3 + A_{21} x_1^2 x_2 + O(|\mu|^2|x| + |x|^4), \end{aligned} \tag{3.9}$$

in which χ_j and A_{jkl} are given by

$$\begin{gathered} A_{30} = \frac{a^7 a_2 f_1'''(0) + a^5 a_1^3 f_2'''(0) + a_1^4 a_2^4 f_3'''(0)}{9a^5 a_1^3 a_2 (a-1)}, \\ A_{21} = \frac{(3a-2)(a^7 a_2 f_1'''(0) + a^5 a_1^3 f_2'''(0) + a_1^4 a_2^4 f_3'''(0))}{9a^5 a_1^3 a_2 (a-1)}. \end{gathered}$$

4. Bifurcation diagrams and computer simulation

In this section, we only give the bifurcation diagrams for (3.9) since it has much richer dynamical behaviors than (3.8) does. Remember that, in this situation, we have $a^5 a_2 f_1''(0) + a^4 a_1^2 f_2''(0) + a_1^3 a_2^3 f_3''(0) = 0$. Noting that $a > 1$ and that, if $a^7 a_2 f_1'''(0) + a^5 a_1^3 f_2'''(0) + a_1^4 a_2^4 f_3'''(0) \neq 0$, then $A_{30} A_{21} > 0$.

Case 1: $A_{30} < 0$ and $A_{21} < 0$. Then under the substitution

$$t \to \frac{A_{21}}{A_{30}} t, \quad x_1 \to -\frac{A_{21}}{\sqrt{|A_{30}|}} x_1, \quad x_2 \to \frac{A_{21}^2}{|A_{30}|^{3/2}} x_2,$$

System (3.9) is transformed into

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \varepsilon_1 x_1 + \varepsilon_2 x_2 - x_1^3 - x_1^2 x_2, \end{aligned} \tag{4.1}$$

where

$$\varepsilon_1 = \Big(\frac{A_{21}}{A_{30}}\Big)^2 \chi_1 = -\frac{2a(3a-2)^2}{a-1}\mu_1,$$

$$\varepsilon_2 = \frac{A_{21}}{A_{30}}\chi_2 = \frac{2(3a-2)}{3a^2(a-1)}(2a\mu_1 + 3(a-1)^2\mu_2).$$

The complete bifurcation diagrams of (4.1) can be found in [8]. Here, we just list two results.

Lemma 4.1. *Let*

$$F^1_+ = \{(\varepsilon_1, \varepsilon_2) : \varepsilon_1 = 0,\ \varepsilon_2 > 0\},$$
$$H^1 = \{(\varepsilon_1, \varepsilon_2) : \varepsilon_2 = 0,\ \varepsilon_1 < 0\},$$
$$H^2 = \{(\varepsilon_1, \varepsilon_2) : \varepsilon_2 = \varepsilon_1,\ \varepsilon_1 > 0\},$$
$$P = \{(\varepsilon_1, \varepsilon_2) : \varepsilon_2 = \frac{4}{5}\varepsilon_1 + o(\varepsilon_1),\ \varepsilon_1 > 0\}.$$
$$K = \{(\varepsilon_1, \varepsilon_2) : \varepsilon_2 = \kappa_0\varepsilon_1 + o(\varepsilon_1),\ \varepsilon_1 > 0\}, \kappa_0 \approx 0.752.$$

For small $\varepsilon_1, \varepsilon_2$, then

(i) if $(\varepsilon_1, \varepsilon_2)$ is in the region between the curves F^1_+ and H^1 or between F^1_+ and H^2, (4.1) *has a limit cycle;*

(ii) if $(\varepsilon_1, \varepsilon_2)$ is in the region between the curves H^2 and P, (4.1) *has three limit cycles: a "big" one and two "small" ones;*

(iii) if $(\varepsilon_1, \varepsilon_2)$ is in the region between the curves P and K, (4.1) *has two limit cycles: the outer one is stable while the inner is unstable.*

Using the expressions of $\varepsilon_1, \varepsilon_2$, we have the following result.

Theorem 4.2. *Suppose that $b = b^* + \mu_1$ and $\tau = \tau^* + \mu_2$. Let*

$$\bar F^1_+ = \{(\mu_1, \mu_2) : \mu_1 = 0,\ \mu_2 > 0\},$$
$$\bar H^1 = \{(\mu_1, \mu_2) : \mu_2 = -\frac{2a}{3(a-1)^2}\mu_1,\ \mu_1 > 0\},$$
$$\bar H^2 = \{(\mu_1, \mu_2) : \mu_2 = -\frac{a(9a-4)}{3(a-1)^2}\mu_1,\ \mu_1 < 0\},$$
$$\bar P = \{(\mu_1, \mu_2) : \mu_2 = -\frac{2a(18a-7)}{15(a-1)^2}\mu_1 + o(|\mu|),\ \mu_1 < 0\}.$$
$$\bar K = \{(\mu_1, \mu_2) : \mu_2 = -\frac{a(\kappa_0(9a-6)+2)}{3(a-1)^2}\mu_1 + o(|\mu_1|),\ \mu_1 < 0\}.$$

For small μ_1, μ_2, then

(i') if (μ_1, μ_2) is in the region between the curves $\bar F^1_+$ and $\bar H^1$ or between $\bar F^1_+$ and $\bar H^2$, (1.1) *has a stable limit cycle;*

(ii') if (μ_1, μ_2) is in the region between the curves H^2 and P, (1.1) *has three limit cycles: a "big" one and two "small" ones;*

(iii') if (μ_1, μ_2) is in the region between the curves P and K, (1.1) *has two limit cycles: the outer one is stable while the inner is unstable.*

Case 2: $A_{30} > 0$ and $A_{21} > 0$. Then under the substitution

$$t \to \frac{A_{21}}{A_{30}}t, \quad x_1 \to \frac{A_{21}}{\sqrt{A_{30}}}x_1, \quad x_2 \to -\frac{A_{21}^2}{A_{30}^{3/2}}x_2,$$

System (3.9) is transformed into

$$
\begin{gathered}
\dot{x}_1 = x_2, \\
\dot{x}_2 = \varepsilon_1 x_1 + \varepsilon_2 x_2 + x_1^3 - x_1^2 x_2,
\end{gathered} \tag{4.2}
$$

where

$$
\begin{gathered}
\varepsilon_1 = \Big(\frac{A_{21}}{A_{30}}\Big)^2 \chi_1 = -\frac{2a(3a-2)^2}{a-1}\mu_1, \\
\varepsilon_2 = -\frac{A_{21}}{A_{30}}\chi_2 = \frac{2(3a-2)}{3a^2(a-1)}(2a\mu_1 + 3(a-1)^2\mu_2).
\end{gathered}
$$

The complete bifurcation diagrams of (4.2) can also be found in [8]. Here, we just list two results.

Lemma 4.3. *for small $\varepsilon_1, \varepsilon_2$, we have:*

(i) System (4.2) *undergoes a Hopf bifurcation for the trivial equilibrium point on the line*

$$
H = \{(\varepsilon_1, \varepsilon_2) : \varepsilon_2 = 0,\ \varepsilon_1 < 0\}.
$$

(ii) On the curve

$$
C = \{(\varepsilon_1, \varepsilon_2) : \varepsilon_2 = -\frac{1}{5}\varepsilon_1 + o(\varepsilon_1),\ \varepsilon_1 < 0\},
$$

(4.2) *undergoes a heteroclinic bifurcation. Moreover, if $(\varepsilon_1, \varepsilon_2)$ is in the region between the curves H and C* (4.2) *has a unique stable periodic orbit.*

Using the expressions of $\varepsilon_1, \varepsilon_2$, we have the following result.

Theorem 4.4. *Suppose that $b = b^* + \mu_1$ and $\tau = \tau^* + \mu_2$. For small μ_1, μ_2, we have:*

(i') System (1.1) *undergoes a Hopf bifurcation for the trivial equilibrium point on the line*

$$
\bar{H} = \{(\mu_1, \mu_2) : \mu_2 = -\frac{2a}{3(a-1)^2}\mu_1,\ \mu_1 > 0\}.
$$

(ii') On the curve

$$
\bar{C} = \{(\mu_1, \mu_2) : \mu_2 = -\frac{a(9a+4)}{15(a-1)^2}\mu_1 + o(\mu_1),\ \mu_1 > 0\},
$$

System (1.1) *undergoes a heteroclinic bifurcation. Moreover, if $(\varepsilon_1, \varepsilon_2)$ is in the region between the curves $\bar{H}$ and $\bar{C}$,* (1.1) *has a unique stable periodic orbit.*

Example 4.5. This example verifies the result in Theorem (4.4)(iii). Let

$$
f_1(x) = 2x + x^3, \quad f_2(x) = x - 2.01x^3, \quad f_3(x) = x + x^3.
$$

Then we have $a_1 = 2$, $a_2 = a_3 = 1$, $a = \sqrt[3]{2}$,and hence

$$
b^* = \frac{\sqrt[3]{2} - 1}{\sqrt[3]{2}}, \quad \tau^* = \frac{1}{\sqrt[3]{2} - 1}.
$$

Since $f_1''(0) = f_2''(0) = f_3''(0) = 0$, we have $A_{20} = A_{11} = 0$ and

$$
A_{30} = -3.030700653228997, \quad A_{21} = -5.393929340342073.
$$

Thus in Theorem 4.2,

$$
\begin{gathered}
\bar{H}^2 = \{(\mu_1, \mu_2) : \mu_2 = -45.623982067834476\mu_1, \mu_1 < 0\}, \\
\bar{P} = \{(\mu_1, \mu_2) : \mu_2 = -38.985746915247205\mu_1, \mu_1 < 0\},
\end{gathered}
$$

$$\bar{K} = \{(\mu_1, \mu_2) : \mu_2 = -37.392570478626254\mu_1, \mu_1 < 0\}.$$

If we choose $(\mu_1, \mu_2) = (-0.0001, 0.0042)$, then it is easy to check that (μ_1, μ_2) is between $\bar{H}^2$ and $\bar{P}$ (Figure 2(a)). According to Theorem 4.2, (1.1) has three limit cycles (Figure 2(b), 2(c) and 2(d)).

If we choose $(\mu_1, \mu_2) = (-0.0001, 0.0038)$, then it is easy to check that (μ_1, μ_2) is between $\bar{P}$ and $\bar{K}$ (Figure 3(a)). According to Theorem 4.2(iii), (1.1) has two "big" limit cycles (Figure 3(b) and 3(c)).

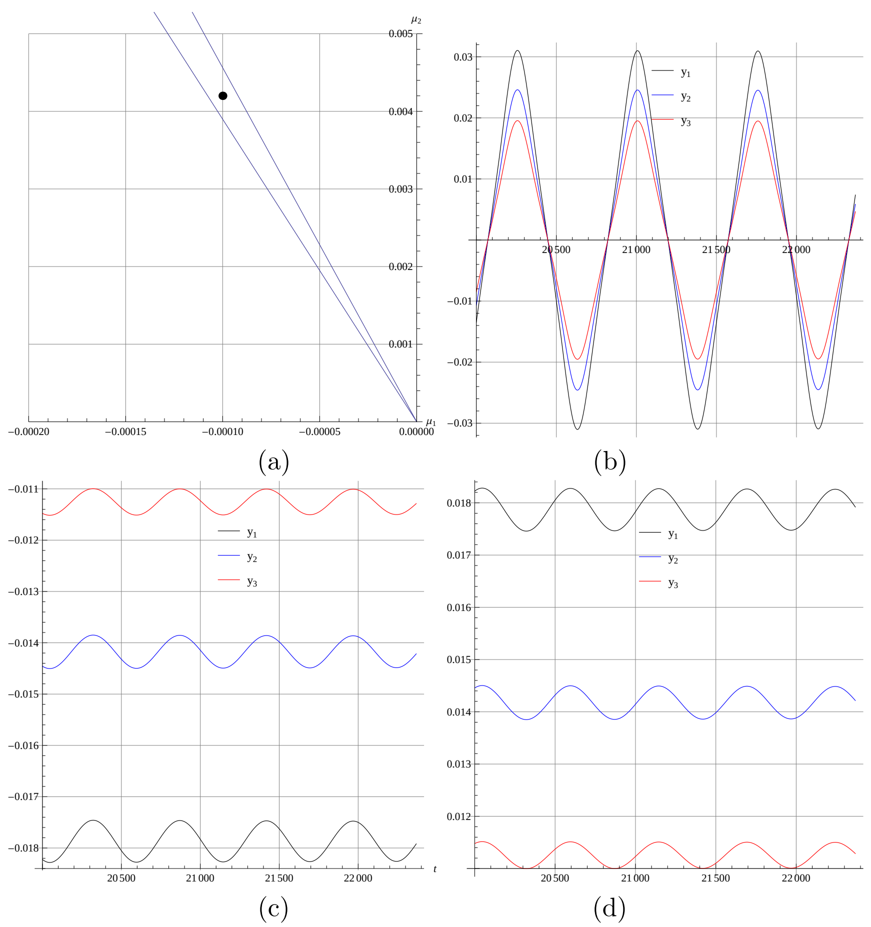

FIGURE 2. (a): (μ_1, μ_2) is between $\bar{H}^2$ and $\bar{P}$; (b): Initial: $y_1(t) = 1, y_2(t) = -0.14, y_3(t) = -0.011$ for $t \leq 0$; (c): Initial: $y_1(t) = 0.0001, y_2(t) = -0.001, y_3(t) = -0.001$ for $t \leq 0$; (d): Initial: $y_1(t) = 0.0178, y_2(t) = 0.01417, y_3(t) = 0.01$ for $t \leq 0$.

Example 4.6. This example verifies the result in Theorem 4.4(ii). Let

$$f_1(x) = \tanh(x), \quad f_2 = 3\tanh(x), \quad f_3(x) = 9\tanh(x).$$

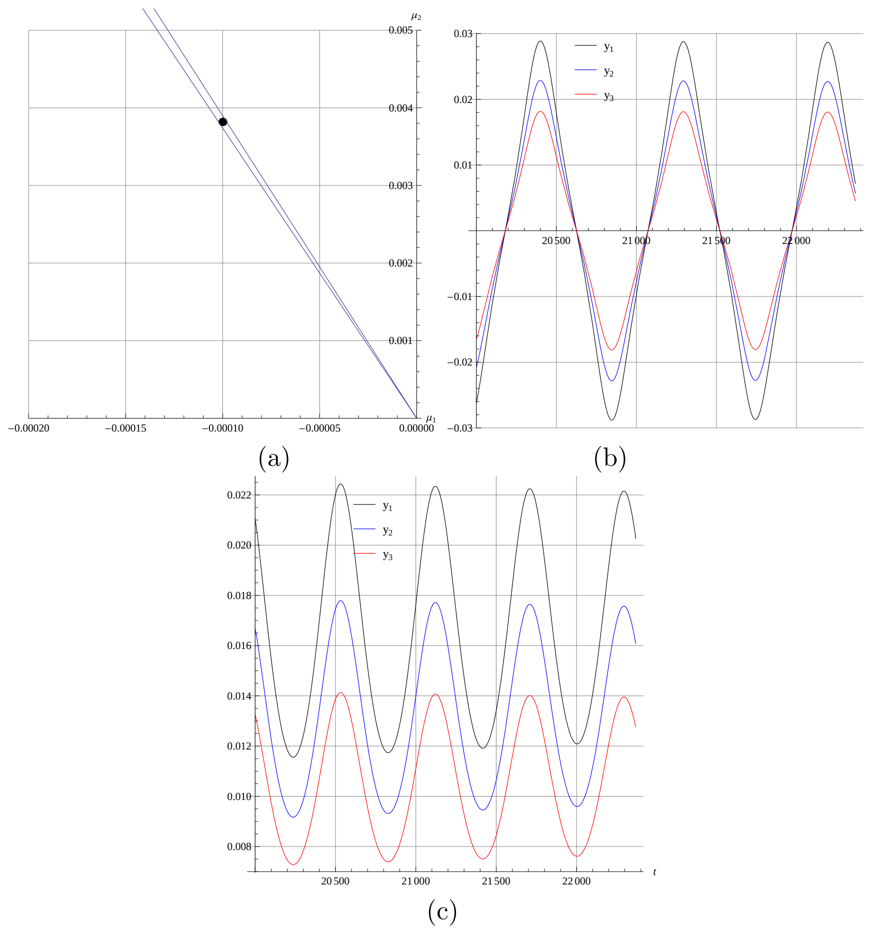

FIGURE 3. $(\mu_1, \mu_2) = (-0.0001, 0.0038)$: there is one periodic limit cycle for $y_1(t) = 0.1, y_2(t) = 0, y_3(t) = 0$ when $t \leq 0$.

Then $a_1 = 1$, $a_2 = 3$, $a_3 = 9$ so that $a = 3$. Thus $b^* = \frac{2}{3}$, $\tau^* = \frac{1}{2}$. Then

$$\bar{H} = \{(\mu_1, \mu_2) : \mu_2 = -\frac{1}{2}\mu_1,\ \mu_1 > 0\},$$

$$\bar{C} = \{(\mu_1, \mu_2) : \mu_2 = \frac{11}{20}\mu_1 + o(\mu_1),\ \mu_1 > 0\}.$$

Choose $\mu_1 = 0.0005, \mu_2 = 0.0000875$ and it is easy to see that $(0.0005, 0.0000875)$ is in the region between the curves $\bar{H}$ and $\bar{C}$. According to Theorem 4.2(ii), (1.1) has a unique stable periodic orbit (see Figure 4).

Conclusion. Neural networks are important both in theory and in application. In this article, we discussed BT singularity of a neural network model and obtained its corresponding normal. Using this normal form, we obtained interesting dynamical behaviors such as Hopf and double limit cycle bifurcations. Two examples were given to verify our theoretical results.

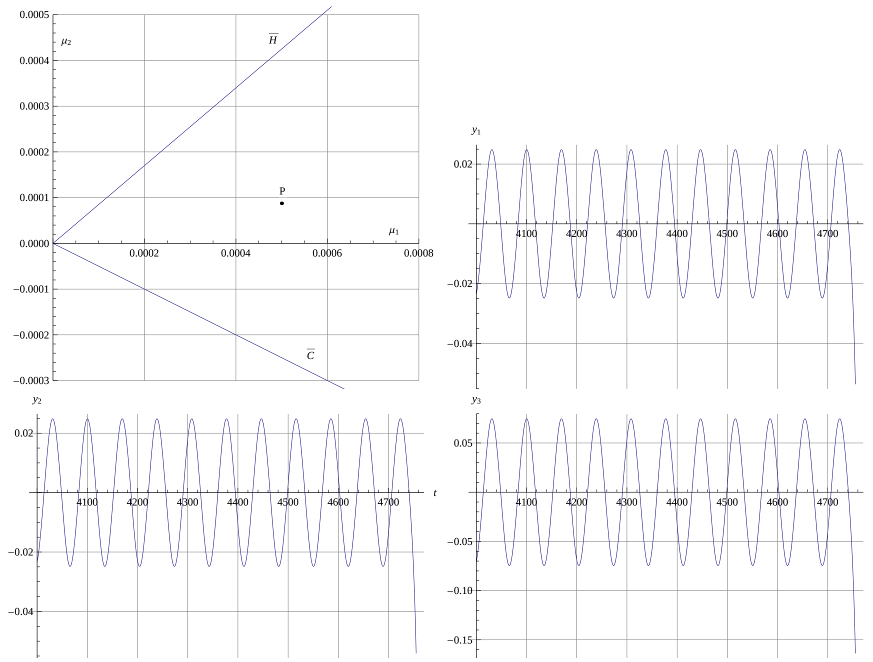

FIGURE 4. When $(\mu_1, \mu_2) = (0.0005, 0.0000875)$ lies between the curves $\bar{H}$ and $\bar{C}$, a periodic solution is bifurcated from the origin.

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XIAOQIN P. WU
DEPARTMENT OF MATHEMATICS, COMPUTER & INFORMATION SCIENCES, MISSISSIPPI VALLEY STATE UNIVERSITY, ITTA BENA, MS 38941, USA
E-mail address: xpwu@mvsu.edu