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CRITICAL DIRICHLET PROBLEMS ON $\mathcal{H}$ DOMAINS OF CARNOT GROUPS

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Dedicated to the memory of our beloved friend Anna

Abstract. The paper deals with the existence of at least one (weak) solution for a wide class of one-parameter subelliptic critical problems in unbounded domains Ω of a Carnot group $\mathbb{G}$, which present several difficulties, due to the intrinsic lack of compactness. More precisely, when the real parameter is sufficiently small, thanks to the celebrated symmetric criticality principle of Palais, we are able to show the existence of at least one nontrivial solution. The proof techniques are based on variational arguments and on a recent compactness result, due to Balogh and Kristály in [2]. In contrast with a persisting assumption in the current literature we do not require any longer the strongly asymptotically contractive condition on the domain Ω. A direct application of the main result in the meaningful subcase of the Heisenberg group is also presented.

1. Introduction

This paper constitutes the initial part of a project devoted to the study of nonlinear equations defined on possibly unbounded domains of Carnot groups. Differential problems involving a subelliptic operator on an unbounded domain Ω of stratified groups have been intensively studied in recent years by many authors, see, among others, the papers of Garofalo and Lanconelli [16], Maad [23, 24], Schindler and Tintarev [32], Tintarev [33] and references therein.

On the contrary, once a domain is not bounded the Folland-Stein space $HW_0^{1,2}(\Omega)$ maybe not be compactly embedded into a Lebesgue space. This lack of compactness produces several difficulties exploiting variational methods. To recover compactness on the unbounded case a persisting hypothesis in the above cited results was the *strongly asymptotically contractive* condition on Ω, introduced by Maad, see [23] for details. Indeed, every bounded domain is strongly asymptotically contractive. In the Euclidean setting unbounded domains were covered in the pioneering paper [12].

Now, we observe that a strongly asymptotically contractive domain Ω is geometrically thin at infinity. In presence of symmetries, by replacing the contractive

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assumption on Ω with a geometrical hypothesis, see condition $(\mathcal{H})$ below, introduced recently by Balogh and Kristály in [2], we are able to treat here subelliptic critical equations, in which the domain is possibly large at infinity.

The purpose of the present paper is to establish the existence of (weak) solutions of the one-parameter problem

$$\begin{gathered} -\Delta_{\mathbb{G}} u + u = h(q) f(u) + \lambda |u|^{2^*-2} u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

More precisely, our strategy is to find a topological group T, acting continuously on $HW_0^{1,2}(\Omega)$, such that the T-invariant closed subspace $HW_{0,T}^{1,2}(\Omega)$ can be compactly embedded in suitable Lebesgue spaces. Successively, assuming the left invariance of the standard Haar measure μ of the Carnot group $\mathbb{G}$, with respect to the action of the group $* : T \times HW_0^{1,2}(\Omega) \to HW_0^{1,2}(\Omega)$, see Bourbaki [6, Chapter III §2 No 4] and Bourbaki [7, Chapter 7 §1 No 1], the principle of symmetric criticality of Palais, see Lemma 3.5 below, can be applied to the associated energy Euler-Lagrange functional I_λ, allowing a variational approach of problem (1.1).

Moreover, as usual, when dealing with critical equations, one of the main difficulties appears since the Palais-Smale condition for the Euler-Lagrange functional I_λ does not hold at any level, but just under a suitable threshold. Along this paper we overcome these difficulties, using some strategies considered in the literature also in context different than the one treated here, see, for instance, papers [5, 22, 28, 30].

Let us briefly introduce the structural setting of problem (1.1). Let $\mathbb{G} = (\mathbb{G}, \circ)$ be a Carnot group of step r and homogeneous dimension $Q > 2$, with neutral element denoted by e. Let $T = (T, \cdot)$ be a closed infinite topological group acting continuously and left-distributively on $\mathbb{G}$ by the map $* : T \times \mathbb{G} \to \mathbb{G}$. Assume that T acts isometrically on the horizontal Folland-Stein space $HW_0^{1,2}(\mathbb{G})$, where the action $\sharp : T \times HW_0^{1,2}(\mathbb{G}) \to HW_0^{1,2}(\mathbb{G})$ is defined for every $(\tau, u) \in T \times HW_0^{1,2}(\mathbb{G})$ by

$$(\tau \sharp u)(q) = u(\tau^{-1} * q) \quad \text{for all } q \in \mathbb{G}.$$

In what follows $d_{CC} : \mathbb{G} \times \mathbb{G} \to \mathbb{R}_0^+$ denotes the Carnot-Carathéodory distance on $\mathbb{G}$, while μ is the natural Haar measure on $\mathbb{G}$ and "$\liminf$" is the Kuratowski lower limit of sets.

Let Ω be a nonempty open T-invariant subset of $\mathbb{G}$, with boundary $\partial\Omega$, and assume that

(H1) for every $(q_k)_k \subset \mathbb{G}$ such that

$$\lim_{k\to\infty} d_{CC}(e, q_k) = \infty \quad \text{and} \quad \mu\Big(\liminf_{k\to\infty}(q_k \circ \Omega)\Big) > 0,$$

where $q_k \circ \Omega = \{q_k \circ q : q \in \Omega\}$, then there exist a subsequence $(q_{k_j})_j$ of $(q_k)_k$ and a sequence of subgroups $(T_{q_{k_j}})_j$ of T, with cardinality $\operatorname{card}(T_{q_{k_j}}) = \infty$, having the property that for all τ_1, $\tau_2 \in T_{q_{k_j}}$, with $\tau_1 \neq \tau_2$, it results

$$\lim_{j\to\infty} \inf_{q\in\mathbb{G}} d_{CC}((\tau_1 * q_{k_j}) \circ q, (\tau_2 * q_{k_j}) \circ q) = \infty.$$

A domain Ω of $\mathbb{G}$, for which condition (H1) holds, is simply called $\mathcal{H}$ *domain*.

In (1.1) the *subelliptic Laplacian operator* $\Delta_{\mathbb{G}}$ on $\mathbb{G}$ is the second-order differential operator

$$\Delta_{\mathbb{G}} = \sum_{k=1}^{m_1} X_k^2,$$

where $\mathcal{B} = \{X_1, \dots, X_{m_1}\}$ is a basis of the first graduated component $\mathfrak{G}_1$ of the stratified Lie algebra $\mathfrak{G} = \oplus_{k=1}^r \mathfrak{G}_k$ associated to $\mathbb{G}$; see Section 2.

The *critical Sobolev exponent* 2^* in the Carnot group $\mathbb{G}$ is $2^* = 2Q/(Q-2)$. The parameter λ is a real number. The nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, with associated primitive

$$F(t) = \int_0^t f(\xi)d\xi \quad \text{for every } t \in \mathbb{R},$$

and satisfies

(H2) $F > 0$ in $\mathbb{R} \setminus \{0\}$, and there exist $C > 0$ and $s \in (1, 2)$ such that

$$|f(t)| \le C|t|^{s-1} \quad \text{for all } t \in \mathbb{R};$$

(H3) there exist $a_0 > 0, \delta > 0$ and $s_1 \in (1, 2)$ such that

$$F(t) \ge a_0|t|^{s_1} \quad \text{for all } t \in \mathbb{R}, \text{ with } |t| \le \delta.$$

Since $Q > 2$, by [18] we know that for all $\varphi \in C_0^\infty(\Omega)$

$$\|\varphi\|_{2^*} \le C_{Q,2}\|D_{\mathbb{G}}\varphi\|_2, \tag{1.2}$$

where $C_{Q,2}$ is a positive constant depending on the dimension Q and

$$D_{\mathbb{G}} = (X_1, \dots, X_{m_1})$$

denotes the horizontal gradient.

Concerning the function h in (1.1), we assume that h satisfies

(H4) $0 \le h \in L^{\frac{2^*}{2^*-s}}(\Omega)$ and there exists a nonempty open set $\Omega_0 \subset \Omega$ such that

$$\inf_{q \in \Omega_0} h(q) > 0.$$

Clearly, condition (H4) simply requires that h be nontrivial and belong to a suitable Lebesgue space. Finally, suppose that

(H5) the functional $\Psi : HW_0^{1,2}(\Omega) \to \mathbb{R}$ given by

$$\Psi(u) = \int_\Omega h(q)f(u)d\mu(q) \quad \text{for all } u \in HW_0^{1,2}(\Omega)$$

is T-invariant, that is $\Psi(\tau\sharp u) = \Psi(u)$ for all $(\tau, u) \in T \times HW_0^{1,2}(\mathbb{G})$.

In Section 2 we present the useful criterion Lemma 2.5 on the validity of assumption (H5). We are now able to state the main existence result for (1.1).

Theorem 1.1. *Let Ω be a $\mathcal{H}$ domain of $\mathbb{G}$. Assume that f and h fulfil* (H2)–(H5). *Then* (1.1) *admits at least one nontrivial solution u_λ in the Folland-Stein space $HW_0^{1,2}(\Omega)$ for all $\lambda \le 0$. Furthermore, if $\lambda > 0$ and h satisfies*

$$\|h\|_{\frac{2^*}{2^*-s}} < \frac{1}{C}\Big(\frac{1}{C_{Q,2}^{(2-s)Q+2s}}\Big(\frac{2^\alpha}{Q}\Big)^{2-s}\Big)^{1/2}, \tag{1.3}$$

where C and s are introduced in (H2), *$C_{Q,2} > 0$ in* (1.2), *and*

$$\alpha = \frac{2^*(6-s)-8}{(2^*-2)(2-s)},$$

then there exists $\lambda^* > 0$ such that problem (1.1) *admits at least one nontrivial solution u_λ in $HW_0^{1,2}(\Omega)$ for all $\lambda \in (0, \lambda^*)$.*

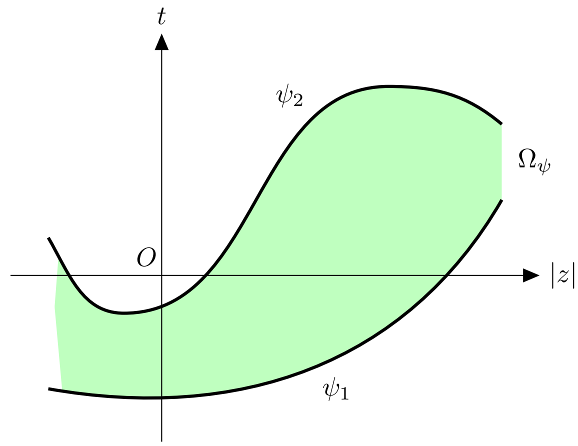

FIGURE 1. A simple prototype of Ω_ψ

Thanks to [2, Theorem 1.1] and Lemma 2.5 below, a direct application of Theorem 1.1 gives the existence of at least one solution for subelliptic equations defined on a special class of (unbounded) domains of the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, $n \geq 1$. More precisely, let ψ_1, $\psi_2 : \mathbb{R}_0^+ \to \mathbb{R}$, $\mathbb{R}_0^+ = [0, \infty)$, be two functions that are bounded on bounded sets, with $\psi_1(t) < \psi_2(t)$ for every $t \in \mathbb{R}_0^+$. Define

$$\Omega_\psi = \big\{q \in \mathbb{H}^n : q = (z,t) \text{ with } \psi_1(|z|) < t < \psi_2(|z|)\big\},$$

where $|z| = \sqrt{\sum_{i=1}^n |z_i|^2}$; see Figure 1.

Then the subelliptic problem (1.1) becomes

$$\begin{gathered} -\Delta_{\mathbb{H}^n} u + u = h(q)f(u) + \lambda |u|^{2^*-2}u \quad \text{in} \quad \Omega_\psi \\ u = 0 \quad \text{on } \partial\Omega_\psi, \end{gathered} \tag{1.4}$$

where $\Delta_{\mathbb{H}^n}$ the subelliptic Kohn-Laplace operator.

Let $\mathbb{U}(n) = U(n) \times \{1\}$, where

$$U(n) = U(n,\mathbb{C}) = \big\{\tau \in GL(n;\mathbb{C}) : \langle \tau z, \tau z'\rangle_{\mathbb{C}^n} = \langle z, z'\rangle_{\mathbb{C}^n} \text{ for all } z, z' \in \mathbb{C}^n\big\},$$

that is $U(n)$ is the usual unitary group. Here $\langle \cdot, \cdot\rangle_{\mathbb{C}^n}$ denotes the standard Hermitian product on $\mathbb{C}^n$, in other words $\langle z, z'\rangle_{\mathbb{C}^n} = \sum_{k=1}^n z_k \cdot \overline{z'_k}$.

Hence, $\mathbb{U}(n)$ is the unitary group endowed with the natural multiplication law $\cdot : \mathbb{U}(n) \times \mathbb{U}(n) \to \mathbb{U}(n)$, which acts continuously and left-distributively on $\mathbb{H}^n$ by the map $* : \mathbb{U}(n) \times \mathbb{H}^n \to \mathbb{H}^n$, defined by

$$\widehat{\tau} * q = (\tau z, t) \quad \text{for all } \widehat{\tau} = (\tau, 1) \in \mathbb{U}(n) \text{ and all } q = (z,t) \in \mathbb{H}^n,$$

thanks to [2, Lemma 3.1]. Taking $T = \mathbb{U}(n)$, then Ω_ψ is $\mathbb{U}(n)$-invariant and a $\mathcal{H}$ domain, as shown in the proof of Theorem 1.1 of [2]. Moreover,

$$HW^{1,2}_{0,\mathbb{U}(n)}(\Omega_\psi) = \{u \in HW^{1,2}_0(\Omega_\psi) : u(z,t) = u(|z|,t) \text{ for all } q = (z,t) \in \Omega_\psi\},$$

that is $HW^{1,2}_{0,\mathbb{U}(n)}(\Omega_\psi) = HW^{1,2}_{\rm cyl}(\Omega_\psi)$ is the space of cylindrically symmetric functions of $HW^{1,2}_0(\Omega_\psi)$.

Finally, $\mathbb{U}(n)$ acts isometrically on the horizontal Folland-Stein space $HW^{1,2}_0(\mathbb{H}^n)$, where the action $\sharp : \mathbb{U}(n) \times HW^{1,2}_0(\mathbb{H}^n) \to HW^{1,2}_0(\mathbb{H}^n)$ is defined for every $(\widehat{\tau}, u)$ in $\mathbb{U}(n) \times HW^{1,2}_0(\mathbb{H}^n)$ by

$$(\widehat{\tau}\sharp u)(q) = u(\tau^{-1}z, t) \quad \text{for all } q = (z,t) \in \mathbb{H}^n,$$

in view of [2, Lemma 3.2] A special case of Theorem 1.1 reads as follows.

Corollary 1.2. *Let Ω_ψ be defined as above. Assume that f and h fulfil* (H2)–(H4)*, and h is cylindrically symmetric, that is $h(q) = h(z,t) = h(|z|,t)$ for every $q = (z,t) \in \Omega_\psi$. Then* (1.1) *admits at least one nontrivial solution u_λ in $HW^{1,2}_{0,\mathbb{U}(n)}(\Omega_\psi)$ for all $\lambda \le 0$.*

Furthermore, if $\lambda > 0$ and h satisfies also (1.3)*, then there exists $\lambda^* > 0$ such that problem* (1.1) *admits at least one nontrivial solution u_λ in $HW^{1,2}_{0,\mathbb{U}(n)}(\Omega_\psi)$ for all $\lambda \in (0, \lambda^*)$.*

If the functions ψ_1 and ψ_2 are bounded, the domain Ω_ψ is strongly asymptotically contractive and the whole space $HW^{1,2}_0(\Omega_\psi)$ is compactly embedded in $L^\nu(\Omega_\psi)$ for every $\nu \in (2, 2^*)$. We refer to [2, 24] for further details. In such a case Corollary 1.2 follows by using the embedding result proved by Garofalo and Lanconelli in [16]. See also Schindler and Tintarev [32].

On the Heisenberg setting, a Rubik-cube technique, see [2], applied to subgroups of $\mathbb{U}(n)$ and suitable variational arguments allow us to obtain further multiplicity results that will be presented in the forthcoming paper [27].

The manuscript is organized as follows. In Section 2 we present the notations and recall some properties of the functional solution space of (1.1). In particular, in order to apply critical point methods to problem (1.1), we need to exploit some analytic properties of the closed subspace $HW^{1,2}_{0,T}(\Omega_\psi)$, introduced above. Then, in the same section, we give the key Lemmas 2.1 and 2.3 which are particularly useful for the proof of Theorem 1.1. Finally, in Section 3 we describe the geometrical profile of the underlying functional in Lemmas 3.1 and 3.2 and we prove the existence result stated in Theorem 1.1.

For general references on the subject and on methods treated along the paper we refer to the monographs [4, 21] as well as [11, 25, 26, 36] and the references therein.

2. Notation and preliminaries

In this section we briefly recall some basic facts on Carnot groups and the functional Folland-Stein space $HW^{1,2}_0(\Omega)$. A Carnot group $\mathbb{G} = (\mathbb{G}, \circ)$ is a connected, simply connected, nilpotent Lie group, whose Lie algebra $\mathfrak{G}$ admits a stratification, i.e.

$$\mathfrak{G} = \oplus_{k=1}^r \mathfrak{G}_k,$$

where the integer r is called the *step* of $\mathbb{G}$, while $\mathfrak{G}_k$ is the linear subspace of finite dimension m_k of $\mathfrak{G}$ for every $k \in \{1, \dots, r\}$, and

$$[\mathfrak{G}_1, \mathfrak{G}_k] = \mathfrak{G}_{k+1} \quad \text{for all } k, \text{ with } 1 \le k < r-1 \quad \text{and} \quad [\mathfrak{G}_1, \mathfrak{G}_r] = \{O\}.$$

In this context the symbol $[\mathfrak{G}_1, \mathfrak{G}_k]$ denotes the subalgebra of $\mathfrak{G}$ generated by the commutators $[X, Y]$, where $X \in \mathfrak{G}_1$ and $Y \in \mathfrak{G}_k$, and where the last bracket denotes the Lie bracket of vector fields, that is $[X, Y] = XY - YX$.

The left translation by $q \in \mathbb{G}$ on $\mathbb{G}$ is given by $\ell_q(p) = q \circ p$ for every $p \in \mathbb{G}$. Let $\Gamma(T\mathbb{G})$ be the space of global sections of the tangent bundle $T\mathbb{G}$ on $\mathbb{G}$. A vector field $X \in \Gamma(T\mathbb{G})$ is left invariant if for every $q \in \mathbb{G}$ one has

$$X(\varphi \circ \ell_q) = (X\varphi) \circ \ell_q,$$

for any $\varphi \in C^\infty(\mathbb{G})$ and $p \in \mathbb{G}$.

The Lie algebra $\mathfrak{G}$ associated to $\mathbb{G}$ is the Lie algebra of left invariant vector fields X on $\mathfrak{G}$. Moreover, $\mathfrak{G}$ is canonically isomorphic to the tangent space $T_e\mathbb{G}$.

Let

$$m = \sum_{k=1}^{r} m_k$$

be the *topological dimension* of the Carnot group $\mathbb{G}$.

The exponential map $\exp_{\mathbb{G}} : \mathfrak{G} \to \mathbb{G}$ is given by $\exp_{\mathbb{G}}(X) = \gamma_X(1)$, where γ_X is the unique integral curve associated to the left invariant vector field X such that $\gamma_X(0) = e$. In other words, the curve γ_X is the unique solution of the Cauchy problem

$$\dot{\gamma}_X(t) = X(\gamma_X(t)), \quad \gamma_X(0) = e. \tag{2.1}$$

The curve γ_X is defined for any $t \in \mathbb{R}$, that is left invariant vector fields are complete. Indeed, $\gamma_X(t+s) = \gamma_X(s)\,\gamma_X(t)$ by (2.1). Therefore, γ_X can be extended in the entire $\mathbb{R}$.

Since $\mathbb{G}$ is nilpotent, connected and simply connected Lie group, the exponential map $\exp_{\mathbb{G}}$ is a smooth diffeomorphism from $\mathfrak{G}$ onto $\mathbb{G}$.

Let $\langle \cdot, \cdot \rangle_0$ be a fixed inner product on the first graduated component $\mathfrak{G}_1$ of $\mathfrak{G}$, with associated orthonormal basis $\mathcal{B} = \{X_1, X_2, \dots, X_{m_1}\}$. From now on, we consider the extension of the inner product $\langle \cdot, \cdot \rangle_0$ to the whole tangent bundle $T\mathbb{G}$ by group translation. The corresponding norm is denoted by $\|\cdot\|_0$. A left invariant vector field $X \in \mathfrak{G}$ is said to be *horizontal* if

$$X(q) \in \operatorname{span}\{X_1(q), \dots, X_{m_1}(q)\}$$

for every $q \in \mathbb{G}$. Indeed, $\mathfrak{G}_1$ is considered to be the *horizontal direction*, while the remaining layers $\mathfrak{G}_2, \cdots, \mathfrak{G}_r$ are viewed as the *vertical directions*. In particular, the last layer $\mathfrak{G}_r$ is the center of the Lie algebra and the horizontal direction G_1 generates in the sense of Lie algebras the whole $\mathfrak{G}$. More precisely,

$$\mathfrak{G}_k = \underbrace{[\mathfrak{G}_1, [\mathfrak{G}_1, [\mathfrak{G}_1, \dots [\mathfrak{G}_1, \mathfrak{G}_1] \cdots]]]}_{k \text{ times}}$$

for all $k = 2, \cdots, r$.

Since the map $\exp_{\mathbb{G}}$ is bijective, for every element $q \in \mathbb{G}$ there exists a unique vector field $X = \sum_{k=1}^{m_1} x_k X_k + \sum_{k=m_1+1}^{m} x_k X_k' \in \mathfrak{G}$ such that

$$q = \exp_{\mathbb{G}}(X) = \exp_{\mathbb{G}}\Big(\sum_{k=1}^{m_1} x_k X_k + \sum_{k=m_1+1}^{m} x_k X_k'\Big),$$

where $\{X_{m_1+1}, \dots, X_m\}$ are non-horizontal vector fields that extend $\mathcal{B}$ to an orthonormal basis $\mathcal{B}^*$ of $\mathfrak{G}$.

Now, observe that $\mathfrak{G} \cong \mathbb{R}^m$. Then, there exists a smooth map ϱ such that the following diagram is commutative

$$\begin{tikzcd}
\mathbb{R}^m \arrow[r, "\pi^{-1}"] \arrow[dr, dashed, "\varrho"'] & \mathfrak{G} = \oplus_{k=1}^{r}\mathfrak{G}_k \arrow[d, "\exp_G"] \\
& \mathbb{G}
\end{tikzcd}$$

where π^{-1} is the inverse of the canonical projection $\pi : \mathfrak{G} \to \mathbb{R}^m$ such that

$$\begin{tikzcd}
\mathbb{R}^m \ni (x_1, \dots, x_{m_1}, \dots, x_m) \arrow[r, "\pi^{-1}"] \arrow[dr, dashed, "\varrho"'] & X = \sum_{k=1}^{m_1} x_k X_k + \sum_{k=m_1+1}^{m} x_k X_k' \in \mathfrak{G} \arrow[d, "\exp_G"] \\
& q \in \mathbb{G}
\end{tikzcd}$$

Thus, we often identify every element $q \in \mathbb{G}$ with its *exponential coordinates* $(x_1, \dots, x_{m_1}, x_{m_1+1}, \dots, x_m) \in \mathbb{R}^m$ respect to the basis $\mathcal{B}^*$ in $\mathfrak{G}$.

More precisely, it is possible to identify the Carnot group $(\mathbb{G}, \circ)$ with $(\mathbb{R}^m, \star)$, where the expression of the group operation $\star$ is given by

$$x \star y = \varrho^{-1}(\varrho(x) \circ \varrho(y)) \quad \text{for all } x,\, y \in \mathbb{R}^m$$

and is explicitly determined by the Baker-Campbell-Hausdorff formula.

Whenever we are in presence of a stratification, it is possible to define a one-parameter group $\{\Delta_\eta\}_{\eta>0}$ of dilatations of the algebra. More precisely, for a fixed real number $\eta > 0$ and all $X \in \mathfrak{G}_k$, we set $\Delta_\eta(X) = \eta^k X$ and extend the map Δ_η to the whole $\mathfrak{G}$ by linearity. Furthermore, the family $\{\Delta_\eta\}_{\eta>0}$ induces a family $\{\delta_\eta\}_{\eta>0}$ of the group automorphisms on $\mathbb{G}$ by the exponential map such that the following diagram is commutative

$$\begin{tikzcd}
\mathbb{G} \arrow[r, "\exp_G^{-1}"] \arrow[d, dashed, "\delta_\eta"] & \mathfrak{G} = \oplus_{k=1}^{r}\mathfrak{G}_k \arrow[d, "\Delta_\eta"] \\
\mathbb{G} & \mathfrak{G} = \oplus_{k=1}^{r}\mathfrak{G}_k \arrow[l, "\exp_G"']
\end{tikzcd}$$

that is

$$\delta_\eta(q) = \exp_{\mathbb{G}}(\Delta_\eta(\exp_{\mathbb{G}}^{-1}(q)))$$

for every $q \in \mathbb{G}$.

The *homogeneous dimension* Q of $\mathbb{G}$, attached to the automorphisms $\{\delta_\eta\}_{\eta>0}$, is defined by

$$Q = \sum_{k=1}^{r} k \dim \mathfrak{G}_k = m_1 + 2m_2 + \dots + r m_r.$$

In particular, the above definition of Q and the fact that $\{\delta_\eta\}_{\eta>0}$ is a family of automorphisms on $\mathbb{G}$ imply that the Jacobian determinant of the dilation δ_η is constant in q and given by η^Q.

Moreover, let μ denote the push-forward of the m-dimensional Lebesgue measure $\mathfrak{L}_m$ on $\mathfrak{G}$ via the exponential map. Then, $d\mu$ defines a biinvariant Haar measure on $\mathbb{G}$ and

$$d\mu(q \circ \delta_\eta) = \eta^Q d\mu(q).$$

Since $\mathbb{G}$ can be identified with $(\mathbb{R}^m, \star)$ by using the exponential map, if $E \subset \mathbb{G}$ is a measurable subset, its Haar measure is explicitly given by $\mu(E) = \mathfrak{L}_m(\rho^{-1}(E))$. Therefore, the same notation will be used for both measures.

Take q_1, $q_2 \in \mathbb{G}$ and let $H\Gamma_{q_1,q_2}(\mathbb{G})$ be the set of piecewise smooth curves γ, such that $\gamma : [0,1] \to \mathbb{G}$, $\dot\gamma(t) \in \mathfrak{G}_1$ a.e. $t \in [0,1]$, $(\gamma(0), \gamma(1)) = (q_1, q_2)$ and

$$\int_0^1 \|\dot\gamma(t)\|_0 dt < \infty.$$

Since $H\Gamma_{q_1,q_2}(\mathbb{G}) \neq \emptyset$ by the celebrated Chow-Rashevskiĭ theorem in [10], it is possible to define the *Carnot-Carathéodory distance* on $\mathbb{G}$, as follows

$$d_{CC}(q_1, q_2) = \inf_{\gamma \in H\Gamma_{q_1,q_2}(\mathbb{G})} \int_0^1 \|\dot\gamma(t)\|_0 dt.$$

The metric d_{CC} is left invariant on $\mathbb{G}$ and for every $\eta > 0$ it results

$$d_{CC}(\delta_\eta(q_1), \delta_\eta(q_2)) = \eta\, d_{CC}(q_1, q_2),$$

for every $q_1, q_2 \in \mathbb{G}$.

The Euclidean distance to the origin $|\cdot|$ on $\mathfrak{G}$ induces a homogeneous pseudo-norm $|\cdot|_{\mathfrak{G}}$ on $\mathfrak{G}$ and (via the exponential map) one on the group $\mathbb{G}$. Indeed, for $X \in \mathfrak{G}$, with $X = \sum_{k=1}^r X_k$, where $X_k \in \mathfrak{G}_k$, define a pseudo-norm on $\mathfrak{G}$ as follows

$$|X|_{\mathfrak{G}} = \Big(\sum_{k=1}^r |X_k|^{2r!/k}\Big)^{2r!}.$$

The induced pseudo-norm on $\mathbb{G}$ has the form

$$|q|_{\mathbb{G}} = |\exp_{\mathbb{G}}^{-1}(q)|_{\mathfrak{G}} \quad \text{for all } q \in \mathbb{G}.$$

The function $|\cdot|_{\mathbb{G}}$ is usually known as the *non-isotropic gauge*. It defines a pseudo-distance on $\mathbb{G}$ given by

$$d(p, q) = |p^{-1} \circ q|_{\mathbb{G}} \quad \text{for all } p, q \in \mathbb{G},$$

that is equivalent to the *Carnot-Carathéodory distance* d_{CC} on $\mathbb{G}$.

Thus, Carnot groups are endowed with the intrinsic Carnot-Carathéodory geometry. The adjective intrinsic is meant to emphasize a privileged role played by the horizontal layer and by group translations and dilations. It is worth stressing that the Carnot-Carathéodory geometry is not Riemannian at any scale. In fact, Carnot groups can be seen as a particular case of more general structures, the so-called *sub-Riemannian spaces*.

The most basic second-order partial differential operator in a Carnot group $\mathbb{G}$ is the *sub-Laplacian*, or equivalently the *horizontal Laplacian in* $\mathbb{G}$, given by

$$\Delta_{\mathbb{G}} = \sum_{k=1}^{m_1} X_k^2.$$

We shall denote by $D_{\mathbb{G}} = (X_1, \dots, X_{m_1})$ the related *horizontal gradient* and set $\|D_{\mathbb{G}} u\|_0 = \big(\sum_{k=1}^{m_1} (X_k u)^2\big)^{1/2}$.

Obviously, Euclidean spaces are commutative Carnot groups, and, more precisely, the only commutative Carnot groups. The simplest example of Carnot group of step two is provided by the Heisenberg group $\mathbb{H}^n$ of topological dimension $m = 2n + 1$ and homogeneous dimension $Q = 2n + 2$, that is the Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$, endowed with the non-Abelian group law

$$q \circ q' = \Big(z + z', t + t' + 2\sum_{i=1}^n (y_i x_i' - x_i y_i')\Big)$$

for all q, $q' \in \mathbb{H}^n$, with

$$q = (z,t) = (x_1, \dots, x_n, y_1, \dots, y_n, t), \quad q' = (z', t') = (x'_1, \dots, x'_n, y'_1, \dots, y'_n, t').$$

The vector fields for $j = 1, \dots, n$

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial t}, \tag{2.2}$$

constitute a basis $\mathcal{B}^*$ for the real Lie algebra $\mathfrak{H} = \mathfrak{G}$ of left invariant vector fields on $\mathbb{H}^n$. The basis $\mathcal{B}^*$ satisfies the Heisenberg canonical commutation relations for position and momentum $[X_j, Y_k] = -4\delta_{jk}\partial/\partial t$, all other commutators being zero.

If $u \in C^2(\mathbb{H}^n)$, then the *horizontal Laplacian in* $\mathbb{H}^n$ of u, called the *Kohn-Spencer Laplacian*, is defined as follows

$$\begin{aligned}
\Delta_{\mathbb{H}^n} u &= \sum_{j=1}^n (X_j^2 + Y_j^2) u \\
&= \sum_{j=1}^n \Big(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial y_j \partial t} \Big) u + 4|z|^2 \frac{\partial^2 u}{\partial t^2},
\end{aligned}$$

and $\Delta_{\mathbb{H}^n}$ is *hypoelliptic* according to the celebrated Theorem 1.1 due to *Hörmander* in [19].

Turning back to (1.1), we need to introduce the suitable solution space. Let Ω be a nontrivial open subset of $\mathbb{G}$. The Folland-Stein horizontal Sobolev space $HW_0^{1,2}(\Omega)$ is the completition of $C_0^\infty(\Omega)$, with respect to the Hilbertian norm

$$\begin{aligned}
\|u\| &= \Big(\int_\Omega \|D_{\mathbb{G}} u\|_0^2 d\mu(q) + \int_\Omega |u|^2 d\mu(q) \Big)^{1/2}, \\
\langle u, \varphi \rangle &= \int_\Omega \langle D_{\mathbb{G}} u, D_{\mathbb{G}} \varphi \rangle_0 \, d\mu(q) + \int_\Omega u\varphi d\mu(q).
\end{aligned} \tag{2.3}$$

Of course, if $\Omega = \mathbb{G}$, then $HW^{1,2}(\mathbb{G}) = HW_0^{1,2}(\mathbb{G})$, where $HW^{1,2}(\mathbb{G})$ denotes the horizontal Sobolev space of the functions $u \in L^2(\mathbb{G})$ such that $D_{\mathbb{G}} u$ exists in the sense of distributions and $\|D_{\mathbb{G}} u\|_0$ is in $L^2(\mathbb{G})$, endowed with the Hilbertian norm (2.3).

In particular, the embedding

$$HW_0^{1,2}(\Omega) \hookrightarrow L^\nu(\Omega) \tag{2.4}$$

is continuous for any $\nu \in [2, 2^*]$; see Folland and Stein [15]. Furthermore, by [17, 20, 35] we know that, if $\mathcal{O}$ is a bounded open set of $\mathbb{G}$, the embedding

$$HW_0^{1,2}(\mathcal{O}) \hookrightarrow\hookrightarrow L^\nu(\mathcal{O}) \tag{2.5}$$

is compact for all ν, with $1 \le \nu < 2^*$.

Let $(\mathbb{G}, \circ)$ be a Carnot group, and $(T, \cdot)$ be a closed topological group, with neutral element $\jmath$. The group T is said to *act continuously* on $\mathbb{G}$, if there exists a map $* : T \times \mathbb{G} \to \mathbb{G}$ such that the following conditions

(H6) $\jmath * q = q$ for every $q \in \mathbb{G}$;
(H7) $\tau_1 * (\tau_2 * q) = (\tau_1 \cdot \tau_2) * q$ for every $\tau_1, \tau_2 \in T$ and $q \in \mathbb{G}$

hold. In addition, the action $*$ is *left distributed* if

(H8) $\tau * (p \circ q) = (\tau * p) \circ (\tau * q)$ for every $\tau \in T$ and $p, q \in \mathbb{G}$.

A set $\Omega \subset \mathbb{G}$ is T-*invariant*, with respect to $*$, if $T * \Omega = \Omega$.

We assume that T induces an action $\sharp : T \times HW_0^{1,2}(\mathbb{G}) \to HW_0^{1,2}(\mathbb{G})$, defined for every $(\tau, u) \in T \times HW_0^{1,2}(\mathbb{G})$ by

$$(\tau\sharp u)(q) = u(\tau^{-1} * q) \quad \text{for all } q \in \mathbb{G}. \tag{2.6}$$

The group T acts *isometrically* on $HW_0^{1,2}(\Omega)$ if

$$\|\tau\sharp u\| = \|u\| \quad \text{for all } (\tau, u) \in T \times HW_0^{1,2}(\mathbb{G}). \tag{2.7}$$

Let

$$HW_{0,T}^{1,2}(\Omega) = \{u \in HW_0^{1,2}(\Omega) : \tau\sharp u = u \text{ for all } \tau \in T\}$$

be the T-invariant subspace of $HW_0^{1,2}(\Omega)$. Clearly, $HW_{0,T}^{1,2}(\Omega)$ is closed, since the action $\sharp$ of T on $HW_0^{1,2}(\Omega)$ is continuous by (H6) and (H7).

The following compactness result is due to Balog and Kristály and given in [2, Theorem 3.1].

Lemma 2.1. *Let $\mathbb{G} = (\mathbb{G}, \circ)$ be a Carnot group of step r and homogeneous dimension $Q > 2$, with neutral element denoted by e. Let $T = (T, \cdot)$ be a closed infinite topological group acting continuously and left distributively on $\mathbb{G}$ by the map $* : T \times \mathbb{G} \to \mathbb{G}$. Assume furthermore that T acts isometrically on $HW_0^{1,2}(\mathbb{G})$, where the action $\sharp : T \times HW_0^{1,2}(\mathbb{G}) \to HW_0^{1,2}(\mathbb{G})$ is defined in* (2.6). *Let Ω be a nonempty T-invariant open subset of $\mathbb{G}$, satisfying condition* (H1). *Then the embedding*

$$HW_{0,T}^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^\nu(\Omega)$$

is compact for every $\nu \in (2, 2^*)$.

Remark 2.2. By (2.4) the embeddings

$$HW_{0,T}^{1,2}(\Omega) \hookrightarrow L^\nu(\Omega)$$

are continuous for every $\nu \in [2, 2^*]$. In particular, there exists a constant C_ν such that

$$\|u\|_\nu \le C_\nu \|u\| \quad \text{for all } u \in HW_{0,T}^{1,p}(\Omega), \tag{2.8}$$

where C_ν depends on ν and Q.

We also notice that inequality (1.2) yields

$$\|u\|_{2^*} \le C_{Q,2} \|D_{\mathbb{H}^n} u\|_2 \tag{2.9}$$

for all $u \in HW_{0,T}^{1,2}(\Omega)$.

Lemma 2.3. *Let $(u_k)_k$ be in $HW_{0,T}^{1,2}(\Omega)$, such that $u_k \rightharpoonup u$ weakly in $HW_{0,T}^{1,2}(\Omega)$, and $u_k \to u$ a.e. in Ω. Then*

$$\begin{gathered}
\lim_{k\to\infty} \int_\Omega |u_k - u|^{2^*} d\mu(q) = \lim_{k\to\infty} \int_\Omega |u_k|^{2^*} d\mu(q) - \int_\Omega |u|^{2^*} d\mu(q), \\
\lim_{k\to\infty} \int_\Omega |u|^{2^*-2} u(u_k - u) d\mu(q) = 0, \\
\lim_{k\to\infty} \int_\Omega |u_k|^{2^*-2} u_k u \, d\mu(q) = \int_\Omega |u|^{2^*} d\mu(q).
\end{gathered} \tag{2.10}$$

Proof. The first part of (2.10) is just the celebrated Brezis-Lieb lemma in [8]. For the second part of (2.10), it is enough to observe that $u_k \rightharpoonup u$ in $L^{2^*}(\Omega)$ by Lemma 2.1 and that $\varphi \mapsto \int_\Omega |u|^{2^*-2}u\varphi d\mu(q)$ is a linear continuous functional on $L^{2^*}(\Omega)$. While the third limit is a consequence of [1, Proposition A.8]. □

A function $u \in HW_0^{1,2}(\Omega)$ is said to be a (weak) *solution* of problem (1.1) if

$$\langle u, \varphi\rangle = \int_\Omega h(q)f(u)\varphi d\mu(q) + \lambda \int_\Omega |u|^{2^*-2}u\varphi d\mu(q) \tag{2.11}$$

for any $\varphi \in HW_0^{1,2}(\Omega)$.

Problem (1.1) has a variational nature and the Euler-Lagrange functional I_λ associated to (1.1) is

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \int_\Omega h(q)F(u)d\mu(q) - \frac{\lambda}{2^*}\int_\Omega |u|^{2^*}d\mu(q).$$

Clearly, the functional I_λ is well-defined in $HW_0^{1,2}(\Omega)$ and, thanks to (H2) and (H4), it is of class $C^1(HW_0^{1,2}(\Omega))$. Moreover, for every $u \in HW_0^{1,2}(\Omega)$

$$\langle I_\lambda'(u), \varphi\rangle = \langle u, \varphi\rangle - \int_\Omega h(q)f(u)\varphi d\mu(q) - \lambda \int_\Omega |u|^{2^*-2}u\varphi d\mu(q) \tag{2.12}$$

for all $\varphi \in HW_0^{1,2}(\Omega)$. Hence, the critical points of I_λ in $HW_0^{1,2}(\Omega)$ are exactly the (weak) solutions of (1.1).

Let $u \in HW_{0,T}^{1,2}(\Omega)$ be a solution of problem (1.1) only in the $HW_{0,T}^{1,2}(\Omega)$ sense, that is

$$\langle u, \varphi\rangle = \int_\Omega h(q)f(u)\varphi d\mu(q) + \lambda \int_\Omega |u|^{2^*-2}u\varphi d\mu(q) \tag{2.13}$$

for any $\varphi \in HW_{0,T}^{1,2}(\Omega)$. Then, $u \in HW_{0,T}^{1,2}(\Omega)$ is a solution of (1.1) in the whole space $HW_0^{1,2}(\Omega)$, that is in sense of definition (2.11), if the *principle of symmetric criticality* of Palais given in [29] holds. To prove this let us recall the well known principle of symmetric criticality of Palais stated in the general form proved in [13] for reflexive strictly convex Banach spaces. For details and comments we refer to [9, Section 5].

More precisely, let $X = (X, \|\cdot\|_X)$ be a reflexive strictly convex Banach space. Suppose that $\mathcal{G}$ is a subgroup of isometries $g : X \to X$, that is g is linear and $\|gu\|_X = \|u\|_X$ for all $u \in X$. Consider the $\mathcal{G}$-invariant closed subspace of X,

$$\Sigma_{\mathcal{G}} = \{u \in X : gu = u \text{ for all } g \in \mathcal{G}\}.$$

By [13, Proposition 3.1] we have

Lemma 2.4. *Let X, $\mathcal{G}$ and Σ be as before and let I be a C^1 functional defined on X such that $I \circ g = I$ for all $g \in \mathcal{G}$. Then $u \in \Sigma_{\mathcal{G}}$ is a critical point of I if and only if u is a critical point of $\mathcal{J} = I|_{\Sigma_{\mathcal{G}}}$.*

From now on we assume that T satisfies the main structural conditions of Theorem 1.1 and that Ω is a nonempty open subset of $\mathbb{G}$, which is T-invariant. We apply the principle of symmetric criticality to the Sobolev space $HW_{0,T}^{1,2}(\Omega)$ under the action $\sharp : T \times HW_0^{1,2}(\mathbb{G}) \to HW_0^{1,2}(\mathbb{G})$ defined in (2.6). Clearly,

$$\|\tau\sharp u\| = \|u\| \quad \text{for all } (\tau, u) \in T \times HW_0^{1,2}(\Omega), \tag{2.14}$$

since T acts isometrically on $HW_0^{1,2}(\Omega)$ by assumption. Moreover, the functional $\Psi : HW_0^{1,2}(\Omega) \to \mathbb{R}$ is T-invariant by assumption (H5). Thus, I_λ is T-invariant in $HW_0^{1,2}(\Omega)$.

Hence, the principle of symmetric criticality of Palais ensures that $u \in HW_{0,T}^{1,2}(\Omega)$ is a solution of problem (1.1) if and only if u is a critical point of the functional $\mathcal{J}_\lambda : HW_{0,T}^{1,2}(\Omega) \to \mathbb{R}$, where $\mathcal{J}_\lambda = I_\lambda|_{HW_{0,T}^{1,2}(\Omega)}$.

We end the section by an essential lemma which shows when the key assumption (H5) is satisfied. To this aim, we need to introduce some facts well known in abstract group measure theory.

Lemma 2.5. *Suppose that the action $*$ of the group T on the Carnot group $\mathbb{G}$ satisfies conditions* (H6)–(H8). *Assume furthermore that the natural Haar measure μ, defined on $\mathbb{G}$, is left $*$ invariant, that is for all measurable subset E of $\mathbb{G}$ and for all $\tau \in T$*

$$\mu(\tau * E) = \mu(E),$$

where $\tau * E = \{\tau * q : q \in E\}$.

If h is T-invariant, that is $h(\tau * q) = h(q)$ for all $\tau \in T$ and $q \in \mathbb{G}$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then (H5) *holds.*

Proof. Fix $\tau \in T$ and $u \in HW_0^{1,2}(\Omega)$. Then, putting $\tau^{-1} * q = p$, we get by (H6)–(H8)

$$\begin{aligned} \Psi(\tau \sharp u) &= \int_\Omega h(q) f((\tau \sharp u)(q)) d\mu(q) = \int_\Omega h(q) f(u(\tau^{-1} * q)) d\mu(q) \\ &= \int_{\tau * \Omega} h(\tau * p) f(u(p)) d\mu(\tau * p) \\ &= \int_\Omega h(p) f(u(p)) d\mu(p) = \Psi(u), \end{aligned}$$

since Ω and h are T-invariant by assumption, and the left $*$ invariance of the measure μ implies

$$d\mu(\tau * p) = d\mu(p) \quad \text{for all } p \in \mathbb{G},$$

which is exactly [7, formula (10)], being 1 the multiplier of μ. See also [3, Chapter 4].

This shows that Ψ is T-invariant, that is Ψ satisfies (H5), and concludes the proof. □

3. Proof of Theorem 1.1

In this section we suppose that the assumptions of Theorem 1.1 are satisfied, without further mentioning. Thus, problem (1.1) has a variational structure and, as explained in Section 2, it is enough to study the critical points of the functional $\mathcal{J}_\lambda : HW_{0,T}^{1,2}(\Omega) \to \mathbb{R}$, defined by

$$\mathcal{J}_\lambda(u) = \frac{1}{2}\|u\|^2 - \int_\Omega h(q) F(u) d\mu(q) - \frac{\lambda}{2^*} \int_\Omega |u|^{2^*} d\mu(q) \tag{3.1}$$

for all $u \in HW_{0,T}^{1,2}(\Omega)$. We first show that $\mathcal{J}_\lambda$ has a useful geometrical profile, and recall that, when $\lambda > 0$, we require also (1.3) on h.

Lemma 3.1. *For any parameter $\lambda \le 1$ there exist positive numbers ρ_0 and $\jmath$ such that $\mathcal{J}_\lambda(u) \ge \jmath$ for any $u \in HW^{1,2}_{0,T}(\Omega)$, with $\|u\| = \rho_0$, and for any function h of the type stated in Theorem 1.1. Moreover,*

$$m_\lambda = \inf_{u \in \overline{B}_{\rho_0}} \mathcal{J}_\lambda(u) < 0,$$

where $B_{\rho_0} = \{u \in HW^{1,2}_{0,T}(\Omega) : \|u\| < \rho_0\}$, and there exist a sequence $(u_k)_k$ in B_{ρ_0} and a function u_λ in $\overline{B}_{\rho_0}$ such that for all k,

$$\begin{gathered}\|u_k\| < \rho_0, \quad m_\lambda \le \mathcal{J}_\lambda(u_k) \le m_\lambda + \frac{1}{k},\\ u_k \rightharpoonup u_\lambda \text{ in } HW^{1,2}_{0,T}(\Omega), \quad u_k \to u_\lambda \text{ a.e. in } \Omega,\\ \mathcal{J}'_\lambda(u_k) \to 0 \quad \text{in } [HW^{1,2}_{0,T}(\Omega)]'.\end{gathered} \tag{3.2}$$

Proof. Fix $\lambda \le 1$. By (H2), Lemma 2.1 and (2.9) we obtain

$$\begin{aligned}\mathcal{J}_\lambda(u) &\ge \frac{1}{2}\|u\|^2 - C\int_\Omega h(q)|u|^s d\mu(q) - \frac{\lambda}{2^*}\|u\|_{2^*}^{2^*}\\ &\ge \frac{1}{2}\|u\|^2 - CC^s_{Q,2}\|h\|_{\frac{2^*}{2^*-s}}\|u\|^s - \frac{\lambda^+}{2^*}C^{2^*}_{Q,2}\|u\|^{2^*},\end{aligned} \tag{3.3}$$

for all $u \in HW^{1,2}_{0,T}(\Omega)$. Therefore, if $\lambda \le 0$, for $\rho_0 > 0$ sufficiently large we have

$$\mathcal{J}_\lambda(u) \ge \rho_0^s\Big[\frac{1}{2}\rho_0^{2-s} - CC^s_{Q,2}\|h\|_{\frac{2^*}{2^*-s}}\Big] = \jmath > 0$$

for all $u \in HW^{1,2}_{0,T}(\Omega)$, with $\|u\| = \rho_0$, since $1 < s < 2$.

In $\lambda \in (0,1]$, then the Young inequality yields for any $\varepsilon > 0$

$$CC^s_{Q,2}\|h\|_{\frac{2^*}{2^*-s}}\|u\|^s \le \varepsilon\|u\|^2 + \varepsilon^{-\frac{s}{2-s}}\Big(CC^s_{Q,2}\|h\|_{\frac{2^*}{2^*-s}}\Big)^{\frac{2}{2-s}},$$

being $1 < s < 2$. Thus, for $\varepsilon = 1/4$ it follows that

$$\mathcal{J}_\lambda(u) \ge \frac{1}{4}\|u\|^2 - \Big(2^s CC^s_{Q,2}\|h\|_{\frac{2^*}{2^*-s}}\Big)^{2/(2-s)} - \frac{C^{2^*}_{Q,2}}{2^*}\|u\|^{2^*},$$

since $0 < \lambda \le 1$. Let us consider the function

$$\eta(t) = \frac{1}{4}t^2 - \frac{C^{2^*}_{Q,2}}{2^*}t^{2^*}, \quad t \ge 0.$$

Now the number $\rho_0 = (2C_{Q,2})^{\frac{1}{2-2^*}} > 0$ is such that

$$\eta(\rho_0) = \max_{t\ge 0}\eta(t) = \frac{1}{2}\Big(\frac{1}{2} - \frac{1}{2^*}\Big)\big(2C^{2^*}_{Q,2}\big)^{2/(2-2^*)} > 0$$

because $2 < 2^*$. Therefore, for any function h, satisfying (1.3), and for any u in $HW^{1,2}_{0,T}(\Omega)$, with $\|u\| = \rho_0$, we obtain

$$\mathcal{J}_\lambda(u) \ge \eta(\rho_0) - \Big(2^s CC^s_{Q,2}\|h\|_{\frac{2^*}{2^*-s}}\Big)^{2/(2-s)} = \jmath > 0,$$

which concludes the proof of the first part.

Let $q_0 \in \Omega_0$ and $R > 0$ be so small that $B \subset \Omega_0$, where $B = B(q_0, 2R)$ is the open ball of center q_0 and radius R and Ω_0 is given in (H4). Choose $\varphi \in C_0^\infty(B)$

such that $0 \leq \varphi \leq 1$, with $\|\varphi\| \leq \rho_0$, and $\int_B \varphi^{s_1} d\mu(q) > 0$. Let $\delta > 0$ be the number given in (H3). For all $t \in (0, \delta)$, then (H3) and (H4) yield

$$\begin{aligned} \mathcal{J}_\lambda(t\varphi) &\leq \frac{1}{2}\|t\varphi\|^2 - \int_\Omega h(q)F(t\varphi)d\mu(q) - \lambda\frac{t^{2^*}}{2^*}\int_\Omega \varphi^{2^*} d\mu(q) \\ &\leq \frac{t^2}{2}\|\varphi\|^2 - \int_\Omega h(q)F(t\varphi)d\mu(q) + \lambda^- \frac{t^{2^*}}{2^*}\int_B \varphi^{2^*} d\mu(q) \\ &\leq \frac{t^2}{2}\rho_0^2 - t^{s_1} a_0 \inf_{q\in\Omega_0} h(q)\int_B \varphi^{s_1} d\mu(q) + \lambda^- \frac{t^{2^*}}{2^*}\int_B \varphi^{2^*} d\mu(q). \end{aligned}$$

Hence, $\mathcal{J}_\lambda(t\varphi) < 0$ for for $t \in (0, \delta)$ sufficiently small, since $1 < s_1 < 2 < 2^*$ by (H3). This shows that $m_\lambda < 0$ and completes the proof.

Applying the Ekeland variational principle in $\overline{B}_{\rho_0}$ and the first part of the lemma, there exists a sequence $(u_k)_k$ in B_{ρ_0} such that

$$m_\lambda \leq \mathcal{J}_\lambda(u_k) \leq m_\lambda + \frac{1}{k}, \quad \mathcal{J}_\lambda(u) \geq \mathcal{J}_\lambda(u_k) - \frac{1}{k}\|u - u_k\|$$

for all $u \in \overline{B}_{\rho_0}$. A standard procedure gives that $\mathcal{J}_\lambda'(u_k) \to 0$ in $[HW^{1,2}_{0,T}(\Omega)]'$ as $k \to \infty$ and, up to a subsequence, the bounded sequence $(u_k)_k \subset B_{\rho_0}$ weakly converges to some $u_\lambda \in \overline{B}_{\rho_0}$ and $u_k \to u_\lambda$ a.e. in Ω. This completes the proof of (3.2) and of the lemma. □

Clearly, (3.2) of Lemma 3.1 implies that the bounded sequence $(u_k)_k$ is a Palais-Smale sequence of $\mathcal{J}_\lambda$ in $HW^{1,2}_{0,T}(\Omega)$ at level m_λ.

Lemma 3.2. *There exists $\lambda^* \in (0, 1]$ such that, up to a subsequence, $(u_k)_k$ strongly converges to some u_λ in $HW^{1,2}_{0,T}(\Omega)$ for all $\lambda < \lambda^*$.*

Proof. Fix $\lambda \leq 1$. By (3.2) of Lemma 3.1, in addition to (2.9) and Lemma 2.3, passing up to a further subsequence, if necessary, $(u_k)_k$ and $u_\lambda \in \overline{B}_{\rho_0}$ satisfy (3.2) and

$$\begin{gathered} u_k \rightharpoonup u_\lambda \text{ in } HW^{1,2}_{0,T}(\Omega), \quad \|u_k\| \to \kappa_\lambda, \\ D_{\mathbb{G}} u_k \rightharpoonup D_{\mathbb{G}} u \quad \text{in } L^2(\Omega, \mathbb{R}^{m_1}), \\ u_k \to u_\lambda \text{ in } L^\nu(\Omega), \quad u_k \to u_\lambda \text{ a.e. in } \Omega, \quad \|u_k - u_\lambda\|_{2^*}^{2^*} \to c_\lambda, \\ |u_k|^{2^*-2}u_k \rightharpoonup |u_\lambda|^{2^*-2}u_\lambda \quad \text{in } L^{2^*/(2^*-1)}(\Omega), \end{gathered} \tag{3.4}$$

where κ_λ and c_λ are nonnegative numbers, and $\nu \in (2, 2^*)$. We claim that

$$\int_\Omega h(q)|u_k - u_\lambda|^s d\mu(q) \to 0. \tag{3.5}$$

Since $h \in L^{\frac{2^*}{2^*-s}}(\Omega)$ and $(u_k)_k$ is bounded in $HW^{1,2}_{0,T}(\Omega)$, by (1.2) for any $\varepsilon > 0$ there exists a measurable set $E \subset \Omega$ such that

$$\begin{aligned} &\int_{\Omega\setminus E} h(q)|u_k - u_\lambda|^s d\mu(q) \\ &\leq \Big(\int_{\Omega\setminus E} |h(q)|^{2^*/(2^*-s)} d\mu(q)\Big)^{(2^*-s)/2^*} \|u_k - u_\lambda\|_{2^*}^2 \leq \frac{\varepsilon}{2}. \end{aligned}$$

Furthermore, for any measurable subset $U \subset E$, by the Hölder inequality

$$\int_U h(q)|u_k - u_\lambda|^s d\mu(q) \leq c\Big(\int_U |h(q)|^{2^*/(2^*-s)} d\mu(q)\Big)^{(2^*-s)/2^*},$$

where $c = \sup_k \|u_k - u_\lambda\|_{2^*}^2$. Hence, $\{h(q)|u_k - u_\lambda|^s\}_k$ is equi-integrable and uniformly bounded in $L^1(E)$, thanks to (H4). Thus by (3.4) and the Vitali convergence theorem, for all $\varepsilon > 0$ there exists $k_0 > 0$ such that

$$\int_E h(q)|u_k - u_\lambda|^s d\mu(q) \le \frac{\varepsilon}{2}$$

for all $k \ge k_0$. Therefore,

$$\int_\Omega h(q)|u_k - u_\lambda|^s d\mu(q) \le \int_{\Omega\setminus E} h(q)|u_k - u_\lambda|^s d\mu(q) + \int_E h(q)|u_k - u_\lambda|^s d\mu(q) \le \varepsilon$$

for all $k \ge k_0$. This proves the claim and (3.5).

Now (H2) and the Hölder inequality give

$$\begin{aligned} \Big|\int_\Omega h(q)f(u_k)(u_k - u_\lambda)d\mu(q)\Big| &\le C\int_\Omega h(q)|u_k|^{s-1}|u_k - u_\lambda|d\mu(q) \\ &\le \tilde{C}\Big(\int_\Omega h(q)|u_k - u_\lambda|^s d\mu(q)\Big)^{1/s}, \end{aligned}$$

for a suitable constant $\tilde{C} > 0$. Thus, by (3.5) it follows that

$$\lim_{k\to\infty}\int_\Omega h(q)f(u_k)(u_k - u_\lambda)d\mu(q) = 0. \tag{3.6}$$

Similarly, by using again (H4) and (H2) we have as $k \to \infty$

$$\int_\Omega h(q)f(u_k)\varphi d\mu(q) \to \int_\Omega h(q)f(u_\lambda)\varphi d\mu(q), \tag{3.7}$$

for any $\varphi \in HW^{1,2}_{0,T}(\Omega)$.

By (3.2), (3.4)–(3.7) we see that u_λ is a solution of (1.1), that is u_λ is a critical point of $\mathcal{J}_\lambda$ in $HW^{1,2}_{0,T}(\Omega)$. In particular, as $k \to \infty$

$$o(1) = \langle \mathcal{J}'_\lambda(u_k) - \mathcal{J}'_\lambda(u_\lambda), u_k - u_\lambda\rangle = (\kappa_\lambda^2 - \|u_\lambda\|^2) - \|u_k\|_{2^*}^{2^*} + \|u_\lambda\|_{2^*}^{2^*} + o(1).$$

Consequently, by (3.4) and the Brézis-Lieb lemma [8] we get the main formula

$$\lim_{k\to\infty}\|u_k - u_\lambda\|^2 = \lambda\lim_{k\to\infty}\|u_k - u_\lambda\|_{2^*}^{2^*} = \lambda\, c_\lambda. \tag{3.8}$$

Let us first consider the case $\lambda \le 0$. Then, (3.8) gives at once that $\|u_k - u_\lambda\| = o(1)$ as $k \to \infty$, that is $(u_k)_k$ strongly converges to u_λ in $HW^{1,2}_{0,T}(\Omega)$, as stated.

Let us now consider the case $\lambda \in (0,1]$. By using (2.9), with $u = u_k - u_\lambda$, we get

$$\lambda\, c_\lambda \ge C_{Q,2}^{2^*} c_\lambda^{2/2^*} \tag{3.9}$$

for all $\lambda \in (0,1]$. Let us define

$$\lambda^* = \begin{cases} \inf\{\lambda \in (0,1] : c_\lambda > 0\}, & \text{if there exists } \lambda \in (0,1] \text{ such that } c_\lambda > 0, \\ 1, & \text{if } c_\lambda = 0 \text{ for all } \lambda \in (0,1]. \end{cases}$$

We claim that $\lambda^* > 0$ if there exists $\lambda > 0$ such that $c_\lambda > 0$. Otherwise, there exists a sequence $(\lambda_k)_k$, with $c_{\lambda_k} > 0$, such that $\lambda_k \to 0$ as $k \to \infty$. Thus, (3.9) implies that

$$\lambda_k c_{\lambda_k}^{1-2/2^*} \ge C_{Q,2}^{2^*} > 0.$$

This is an obvious contradiction since $\{c_\lambda\}_{\lambda\in(0,1]}$ is uniformly bounded above by (2.9). Indeed, $(u_k)_k \subset B_{\rho_0}$, $u_\lambda \in \overline{B}_{\rho_0}$ and ρ_0, given in Lemma 3.1, is independent of λ.

Hence, $c_\lambda = 0$ for any $\lambda \in (0, \lambda^*)$. Therefore, for all $\lambda \in (0, \lambda^*)$,
$$\lim_{k\to\infty} \|u_k - u_\lambda\|_{2^*} = 0.$$
Now (3.8) implies
$$\lim_{k\to\infty} \|u_k - u_\lambda\| = 0.$$
In conclusion, $u_k \to u_\lambda$ as $k \to \infty$ in $HW^{1,2}_{0,T}(\Omega)$ for all $\lambda < \lambda^*$, as stated. $\square$

Proof of Theorem 1.1. Let $\mathcal{J}_\lambda$ be the restriction of the energy functional I_λ to the subspace $HW^{1,2}_{0,T}(\Omega)$. For any $\lambda \le 1$ Lemma 3.1 and the Ekeland variational principle give the existence of a Palais-Smale sequence $(u_k)_k$ in $HW^{1,2}_{0,T}(\Omega)$ of $\mathcal{J}_\lambda$ at level $m_\lambda < 0$. Moreover, by Lemma 3.2 there exists $\lambda^* > 0$ such that, up to a subsequence, $(u_k)_k$ strongly converges to some u_λ in $HW^{1,2}_{0,T}(\Omega)$ for all $\lambda < \lambda^*$. Furthermore, $m_\lambda = \mathcal{J}_\lambda(u_\lambda) < 0$ and $\mathcal{J}'_\lambda(u_\lambda) = 0$ for all $\lambda < \lambda^*$. Consequently, the function $u_\lambda \in HW^{1,2}_{0,T}(\Omega)$ is a nontrivial critical point of the functional $\mathcal{J}_\lambda$. Now, as observed in Section 2, since the action $\sharp : T \times HW^{1,2}_0(\Omega) \to HW^{1,2}_0(\Omega)$ given in (2.6) is supposed to be isometric, the functional I_λ is T-invariant by assumption (H5). Hence, the principle of symmetric criticality of Palais, recalled in (2.4), implies that $u_\lambda \in HW^{1,2}_{0,T}(\Omega)$ is a nontrivial critical point also for I_λ in $HW^{1,2}_0(\Omega)$, that is a nontrivial solution for (1.1) in the sense of definition (2.11). This completes the proof. $\square$

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