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LOWER BOUNDS ON THE FUNDAMENTAL SPECTRAL GAP WITH ROBIN BOUNDARY CONDITIONS

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Abstract. This article investigates the gap between the first two eigenvalues of Schrödinger operators on an interval subjected to the Robin and Neumann boundary conditions for a class of linear convex potentials. Furthermore, when the potential is constant the gap is minimized. Meanwhile, we establish a link between the first eigenvalues and the real roots of the first derivative of the Airy functions Ai′ and Bi′.

1. Introduction

We consider the low-lying eigenvalues λ of a self-adjoint Schrödinger problem on an interval

$$Hu := -\frac{d^2u}{dx^2} + q(x)u = \lambda u, \quad x \in [0, \pi]. \tag{1.1}$$

The restriction to an interval of length π is merely a convenient normalization.

General bounds on the gap between the first two eigenvalues $\Gamma := \lambda_2 - \lambda_1$ of Schrödinger operators have gained considerable attention during the last decades. The quantity Γ has various physical and mathematical applications. For example, in quantum mechanics, the fundamental gap represents the energy needed to achieve the first excited state from the ground state of the particle described by the considered Schrödinger operators, the so called excitation energy. The fundamental gap is also of importance in quantum field theory and statistical mechanics. To give an example, Van den Berg in [8] applied gap results of the Laplace operator to provide sufficient conditions for a free boson gas to fill the ground state alone under the thermodynamic limit macroscopically.

In numerical mathematics, the fundamental gap is used to determine the convergence rate of numerical computation methods, such as discretization of the finite element method, which involves the approximation of differential operators by matrices. In this setting, the difference between the first two eigenvalues represents the ability to determine the first eigenvalue and eigenvector. Probabilistically, the fundamental gap controls the asymptotic exponential rate of convergence to equilibrium for the associated Markovian semigroup of the considered Schrödinger operator, and its related to the log-Sobolev constant (see [7, 20]).

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or the problem (1.1) with Dirichlet boundary conditions, Ashbaugh and Benguria [5], established that the optimal lower bound for Γ for symmetric single-well potentials is reached if and only if q is constant on $(0,\pi)$. Lavine [17] investigated the class of convex potentials on $[0,\pi]$ and demonstrated with either Dirichlet or Neumann boundary conditions, that the constant potential function minimizes Γ. Later Horváth [12] returned to the problem of single-well potentials using Lavine's methods, but without making any symmetry assumptions, and proved that the constant potential was optimal with some restrictions on the transition point, and in 2015 Yu and Yang [23] extended Horváth's result by allowing other transition points and both Dirichlet and Neumann boundary conditions. Recently Harrell and El Allali [9] used direct optimization methods to prove sharp lower bounds for Γ with general single-well potential $q(x)$, without any restriction on the transition point $a \in [0,\pi]$ and found similar results in the case where the potential is convex. Additionally, Harrell and El Allali analyzed the case where $q = q_0 + q_1$, where q_0 is a fixed background potential energy, and q_1 is assumed either single-well or convex. In contrast to the previous studies of single-well potentials, which restrict the transition point in some way, the minimizing potentials they found are in general step functions and *not* necessarily constant unless additional criteria are imposed. In the classic case where $p = 1$ they retrieved Lavine's with different arguments the result of Lavine that Γ is uniquely minimized among convex q by the constant and in the case of single-well potentials, with no restrictions on the position of the minimum, they proved the innovative lower bound $\Gamma \geq 2.04575\dots$

In higher dimensions, the first lower bound on the spectral gap was made by Payne and Weinberger [19] who proved the spectral gap of the Neumann Laplacian on a bounded domain Ω is bounded from below by $\frac{\pi^2}{D^2}$, where D is the diameter of Ω. Later, Andrews and Clutterbuck [3] showed that Dirichlet Schrödinger operators on a convex domain with a convex potential have a fundamental spectral gap that is always greater than $\frac{3\pi^2}{D^2}$. Smits [20] investigated the topic of the lower bounds on the fundamental gap under Robin boundary conditions, which lies between the Dirichlet and Neumann cases. There is an extensive literature on extending these gap results for Neumann and Dirichlet boundary conditions to the case of Robin boundary conditions, see for example Laugesen [16], Andrews, Clutterbuck, and Hauer [2], Chapter 4 (by Bucur, Freitas, and Kennedy) of the book [10], Kielty [15]. For more information about the history of the fundamental gap see [4, 14].

Throughout this article, we consider the following problem under Robin boundary conditions.

$$\begin{aligned} -u'' + q(x)u = \lambda u \quad x \in [0,\pi], \\ u'(0) = -\eta u(0), \\ u'(\pi) = \zeta u(\pi). \end{aligned} \tag{1.2}$$

The Robin boundary conditions have the physical interpretation of radiation if $\eta < 0$ and $\zeta > 0$, absorption when $\eta > 0$ and $\zeta < 0$, and insulation when $\eta = 0$ and $\zeta = 0$. We briefly recall the recent literature concerning related works of problem (1.2) subject to the Robin boundary conditions. Andrews, Clutterbuck and Hauer [1] proved that the minimizer of $\Gamma[q]$ is the constant potential with either convex or single-well potentials. Ashbaugh and Kielty [6] also established that the fundamental gap is an increasing function of the Robin parameters for the class of convex symmetric potentials. Motivated by these results, we shall prove in this paper that the optimal lower bound of $\Gamma[q]$ among convex potentials is the constant potential for two different Robin parameters of the problem (1.2), which is regarded as an open problem of the authors Ashbaugh and Kielty , (in [6], see open problem 3.1).

The proof of our main result will be based on the recent work by Andrews, Clutterbuck and Hauer [1] and some techniques used in Master thesis of Höltschl [11] which has not appeared in archival journals in the case of linear potential.

This work is arranged as follows: in section 2, we derive some simple properties of the fundamental gap Γ. In section 3, we shall study the optimal estimates of the fundamental gap Γ for Schrödinger operators with Robin boundary conditions. In section 4, we establish a relation between the first eigenvalues of Neumann Schrödinger operators on an interval and the real roots of the first derivative of the Airy functions Ai' and Bi', where $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are the Airy functions on the bounded solution of the ordinary differential equation

$$u"(x) = xu(x).$$

2. Preliminaries and basics

Definition 2.1. The fundamental gap which is denoted by $\Gamma[q]$, is the difference between the first two eigenvalues

$$\Gamma[q] = \lambda_2(q) - \lambda_1(q).$$

The fundamental gap for the Schrödinger problem with Robin boundary conditions depends on the potential and the length of the underlying interval and the parameters η, ζ.

By definition we remark that the fundamental gap of a Schrödinger problem with Robin boundary conditions on an interval $[0, \pi]$ unaffected by adding a constant $\Gamma[q] = \Gamma[q+c]$. As a consequence of this remark, which will be very important later on, if we have to deal with linear potentials, we only need to consider potentials of the form $q(x) = tx$ with $t > 0$.

Theorem 2.2 ([22])**.** *The spectrum of Schrödinger problem with Robin boundary conditions has a discrete spectrum of simple eigenvalues, satisfies*

$$\lambda_1 < \lambda_2 < \dots < \lambda_n \to \infty.$$

Moreover, the eigenfunction u_n corresponding to the eigenvalue λ_n has exactly $n-1$ zeros in $(0, \pi)$.

Now, we give the Hellmann-Feynmann result [13] for the variation of eigenvalues with respect to a family of potentials of the problem (1.2) under Robin boundary conditions.

Lemma 2.3. *Suppose that $q(.,t)$ is a one-parameter family of real-valued, locally L^1 function with $\frac{\partial q}{\partial t}(x,t) \in L^1(0,\pi)$ and $\inf q(x,t) > -\infty$. Then*

$$\frac{d\lambda_n(t)}{dt} = \int_0^\pi \frac{\partial q}{\partial t}(x,t) u_n^2(x,t)\, dx.$$

Proof. Denote $\dot{u} = \frac{du}{dt}$, the potential q depends integrably on x and differentiably on t. We define $R_t : H^1(0,\pi) \setminus \{0\} \to \mathbb{R}$, the Rayleigh quotient by

$$R_t(u) = \frac{\int_0^\pi (u')^2 dx + \int_0^\pi q u^2 dx - (\eta u^2(0) + \zeta u^2(\pi))}{\int_0^\pi u^2 dx}$$

and we consider the problem (1.2) with Robin boundary conditions then

$$\begin{aligned}\frac{d\lambda_n(t)}{dt} &= \frac{1}{\int_0^\pi u_n^2 dx}\Big(\int_0^\pi (2u_n'\dot{u}_n' + \dot{q}u_n^2 + 2u_n\dot{u}_n q)dx \\ &\quad - (2\eta u_n(0)\dot{u}_n(0) + 2\zeta u_n(\pi)\dot{u}_n(\pi))\Big) \\ &\quad - \frac{1}{\big(\int_0^\pi u_n^2 dx\big)^2}\Big(\int_0^\pi (u_n')^2 dx + \int_0^\pi q u_n^2 dx - (\eta u_n^2(0) + \zeta u_n^2(\pi)\Big) \\ &\quad \times \Big(\int_0^\pi 2u_n\dot{u}_n dx\Big).\end{aligned}$$

Integrating by parts twice

$$\int_0^\pi 2u_n'\dot{u}_n' dx = 2\int_0^\pi \lambda_n u_n \dot{u}_n dx - 2\int_0^\pi q y_n \dot{u}_n dx + 2(\eta u_n(0)\dot{u}_n(0) + \zeta u_n(\pi)\dot{u}_n(\pi)).$$

So

$$\frac{d\lambda_n(t)}{dt} = \frac{1}{\int_0^\pi u_n^2 dx}\Big(2\int_0^\pi \lambda_n u_n\dot{u}_n dx + \int_0^\pi \dot{q}u^2 dx\Big) - \frac{\lambda_n}{\int_0^\pi u_n^2 dx}\int_0^\pi 2u_n\dot{u}_n dx.$$

Then

$$\frac{d\lambda_n(t)}{dt} = \frac{1}{\int_0^\pi u_n^2 dx}\int_0^\pi \dot{q}u_n^2 dx.$$

Noting that $\int_0^\pi u_n^2 dx = 1$. So

$$\frac{d\lambda_n(t)}{dt} = \int_0^\pi \frac{\partial q}{\partial t}(x,t)u_n^2(x,t)dx.$$

□

3. Characterization of optimizers

In this section we prove that the minimizer of $\Gamma[q]$ among convex potentials q is the constant potential. The proof is based on refined arguments from Lavine's proof of the fundamental gap conjecture with Robin boundary condition.

Lemma 3.1. *Consider problem* (1.2) *with Robin boundary conditions such that* $q(x) = tx$. *Then for every* y *satisfying* (1.2), *we have*

$$\pi((u'(\pi))^2 + (\lambda - t\pi)u^2(\pi)) = \int_0^\pi (2\lambda - 3tx)u^2(x)dx + \eta u^2(0) + \zeta u^2(\pi), \tag{3.1}$$

$$\begin{aligned}&\pi((u'(\pi))^2 + (\lambda - t\pi)u^2(\pi)) \\ &= \frac{1}{\pi}\int_0^\pi x(4\lambda - 5tx)u^2(x)dx - \frac{1}{\pi}\big(u^2(\pi) - u^2(0)\big) + 2\zeta u^2(\pi).\end{aligned} \tag{3.2}$$

Proof. We have

$$\begin{aligned}\pi((u'(\pi))^2 + (\lambda - t\pi)u^2(\pi)) &= \int_0^\pi \frac{d}{dx}x\big((u'(x))^2 + (\lambda - tx)u^2(x)\big)dx \\ &= \int_0^\pi (u'(\pi))^2 + (\lambda - tx)u^2(x))dx + \int_0^\pi x(2u'(x)u''(x) \\ &\quad + 2u(x)u'(x)(\lambda - tx) - tu^2(x))dx \\ &= \int_0^\pi ((u'(\pi))^2 + (\lambda - tx)u^2(x))dx - \int_0^\pi txu^2(x)dx.\end{aligned}$$

Since u satisfies problem (1.2), it follows that

$$\int_0^\pi (u'(x))^2 dx - \int_0^\pi (\lambda - tx)u^2(x)dx = \int_0^\pi (u(x)u'(x))'dx = \eta u^2(0) + \zeta u^2(\pi).$$

So

$$-\int_0^\pi (u'(x))^2 dx + \int_0^\pi (\lambda - tx)u^2(x)dx + \eta u^2(0) + \zeta u^2(\pi) = 0.$$

Thus,

$$\pi((u'(\pi))^2 + (\lambda - t\pi)u^2(\pi)) = \int_0^\pi (2\lambda - 3tx)u^2(x)dx + \eta u^2(0) + \zeta u^2(\pi).$$

In the same way as formula (3.1), we can prove (3.2). $\square$

Theorem 3.2. *Consider problem* (1.2) *with Robin boundary conditions. Then for every convex and non-affine potential q, there exists a linear potential $q_t = tx$ such that*

$$\Gamma[q] \geq \Gamma[q_t].$$

Proof. Let q be a convex non affine potential and $L_q(x) = tx + b$ the linear potential such that $q(x_\pm) = L_q(x_\pm)$. We know by [6] that there exist $0 \leq x_- < x_+ \leq \pi$ and

$$\begin{gathered} u_2^2(x) \geq u_1^2(x) \quad \text{on } (0, x_-) \cup (x_+, \pi), \\ u_1^2(x) > u_2^2(x) \quad \text{on } (x_-, x_+). \end{gathered}$$

By convexity of q,

$$\begin{gathered} q - L_q \geq 0 \quad \text{on } (0, x_-) \cup (x_+, \pi), \\ q - L_q \leq 0 \quad \text{on } (x_-, x_+). \end{gathered}$$

So

$$\int_0^\pi (q - L_q)(u_2^2 - u_1^2)dx > 0.$$

Let

$$L_q(\theta) = \theta q + (1-\theta)L_q \quad \text{for } \theta \in (0,1).$$

Therefore $\dot{L_q}(\theta) = q - L_q$. Then

$$\frac{d\Gamma(L_q(\theta))}{d\theta} = \int_0^\pi (q - L_q)(u_2^2 - u_1^2)dx > 0.$$

Integrating this inequality with respect to θ over $(0,1)$, we find that

$$\int_0^1 \frac{d\Gamma(L_q(\theta))}{d\theta} = \Gamma[q] - \Gamma[L_q] \geq 0.$$

Then $\Gamma[q] \geq \Gamma[L_q] = \Gamma[q_t]$. $\square$

Theorem 3.3. *Consider problem* (1.2) *with Robin boundary conditions. Then for every convex and non-affine potential q we have*

$$\Gamma[q] \geq \Gamma[0]$$

if and only if $\eta > 0$ and $\zeta < 0$.

Proof. We consider a family of potentials $q^\theta = \theta tx$. From Lemma 2.3 we have

$$\frac{d(\lambda_2(q^\theta) - \lambda_1(q^\theta))}{d\theta} = t\int_0^\pi x(u_2^2(x) - u_1^2(x))dx.$$

Suppose that the critical point of the gap is achieved at some $t \neq 0$, then

$$\int_0^\pi x(u_2^2(x) - u_1^2(x))dx = 0.$$

By Lemma 3.1 we find that

$$\begin{aligned}
&\int_0^\pi (2\lambda - 3tx)u^2(x)dx + \eta u^2(0) + \zeta u^2(\pi) \\
&= \frac{1}{\pi}\int_0^\pi x(4\lambda - 5tx)u^2(x)dx - \frac{1}{\pi}(u^2(\pi) - u^2(0)) + 2\zeta u^2(\pi).
\end{aligned}$$

Then

$$\begin{aligned}
&2\lambda_n + \zeta u_n^2(\pi) + \eta u_n^2(0) - 2\zeta u_n^2(\pi) + \frac{1}{\pi}(u_n^2(\pi) - u_n^2(0)) \\
&= \frac{-5t}{\pi}\int_0^\pi x^2u_n^2(x)dx + \Big(3t + \frac{4\lambda_n}{\pi}\Big)\int_0^\pi xu_n^2(x)dx.
\end{aligned}$$

Since

$$\int_0^\pi x(u_2^2(x) - u_1^2(x))dx = 0,$$

it follows that

$$\begin{aligned}
&2(\lambda_2 - \lambda_1) - \zeta(u_2^2(\pi) - u_1^2(\pi)) + \eta(u_2^2(0) - u_1^2(0)) \\
&+ \frac{1}{\pi}(u_2^2(\pi) - u_1^2(\pi) - u_2^2(0) + u_1^2(0)) \\
&= \frac{-5t}{\pi}\int_0^\pi x^2(u_2^2(x) - u_1^2(x))dx \\
&= \frac{-5t}{\pi}\int_0^\pi (x^2 - Ax - B)(u_2^2(x) - u_1^2(x))dx < 0 \quad \text{if } t > 0,
\end{aligned}$$

choosing A and B. Which yields a contradiction with $\eta > 0$ and $\zeta < 0$ and $(u_2^2(\pi)-u_1^2(\pi))-(u_2^2(0)-u_1^2(0)) > 0$. Then the fundamental gap for linear potentials $q_t(x) = tx$ achieves its minimum at $t = 0$. □

Example and numerical simulation. By Theorem 3.3, we have $\Gamma[q] \geq \Gamma[0]$. The eigenfunctions of problem (1.2) for $t = 0$ are given by

$$u(x) = c_1\sin(\sqrt{\lambda}x) + c_2\cos(\sqrt{\lambda}x).$$

The Robin boundary conditions

$$\begin{gathered}
u'(0) = -\eta u(0), \\
u'(\pi) = \zeta u(\pi),
\end{gathered} \tag{3.3}$$

give

$$\tan(k\pi)\Big(\frac{\eta\zeta}{k} - k\Big) = \eta + \zeta$$

with $k = \sqrt{\lambda}$. Using Mathematica, we can calculate the approximates of non-negative real roots of the transcendental equation $\tan(k\pi)(\frac{\eta\zeta}{k} - k) = \eta + \zeta$. For simplicity we fix $\zeta = -1$, obtaining the results in Table 1. Then for $\eta \to 0$, we have $\Gamma[q] \geq 1.335526\dots$.

Table 1.

η	λ_1	λ_2	Γ
0.01	0.142009	1.479635	1.337626
0.1	0.092975	1.428501	1.335526
10	0.654731	2.964465	2.309734
100	0.623687	2.810751	2.187064
1000	0.620702	2.795907	2.175205

4. Estimates on the fundamental gap for linear potentials

In this section, we establish a relation between the first two eigenvalues of the Neumann Schrödinger operators and the real roots of the first derivative of the Airy functions Ai and Bi. We take the Robin boundary conditions and sending $\eta \to 0$, $\zeta \to 0$ so, we recover the Neumann boundary conditions.

We denote by $\Lambda = \bigcup_{n\in\mathbb{N}} \alpha'_n$ the set of zeros of Ai' and by $\vee = \bigcup_{n\in\mathbb{N}} \beta'_n$ the set of zeros of Bi'. For $x \in \mathbb{R} \setminus \Lambda$, we introduce the function

$$f(x) = \frac{\mathrm{Bi}'(x)}{\mathrm{Ai}'(x)}.$$

The zeros β'_n of Bi' coincide with the zeros of f, while the zeros α'_n of Ai' give the singularities of f. We call $f_{/(\beta'_{n+1},\beta'_n)}$ the $n+1$-th branch of f and $f_{/(\beta'_1,0)}$ the first branch of f.

Lemma 4.1. *The function f is strictly increasing on every branch.*

Proof. Differentiate f we obtain

$$f'(x) = \frac{\mathrm{Bi}''(x)\mathrm{Ai}'(x) - \mathrm{Bi}'(x)\mathrm{Ai}''(x)}{\mathrm{Ai}'(x)^2},$$
$$f'(x) = \frac{-xW(\mathrm{Ai}(x), \mathrm{Bi}(x))}{\mathrm{Ai}'(x)^2}.$$

The Wronskian of Ai and Bi is known to be π^{-1}. Thus we obtain

$$f'(x) = \frac{-x\pi^{-1}}{\mathrm{Ai}'(x)^2} > 0 \quad \text{for each } x < 0.$$

Let $D : (-\infty, \alpha'_1) \to (0, \infty)$ be the function describing the distance of two consecutive branches of f, defined by $D(\alpha'_i) = \alpha'_{i-1} - \alpha'_i$ such that D is continuous on $(-\infty, \alpha'_1)$. $\square$

Theorem 4.2. *Consider the Neumann Schrödinger problem with linear potential $q(x) = tx$ with $t > 0$. Let α'_n denote the n-th zero of Ai'. Then*

$$\Gamma[q] \geq (\alpha'_1 - \alpha'_2)t^{2/3}.$$

Proof. Consider the Schrödinger equation

$$-u''(x) + q(x)u(x) = \lambda u(x).$$

Using the change of variable $\varepsilon = \lambda/t^{2/3}$ and $x = z/t^{1/3}$, we obtain the new equation

$$u''(z) = (z - \varepsilon)u(z).$$

The eigenfunctions of Schrödinger equation with linear potentials have the form

$$u(z) = c_1\mathrm{Ai}(z-\varepsilon) + c_2\mathrm{Bi}(z-\varepsilon)$$

with $c_1, c_2 \in \mathbb{R}$.

$$u(x) = c_1\mathrm{Ai}(t^{1/3}x - \lambda t^{-2/3}) + c_2\mathrm{Bi}(t^{1/3}x - \lambda t^{-2/3}).$$

Applying Neumann boundary conditions at the endpoints $u'(0) = u'(\pi) = 0$, we find that

$$\mathrm{Bi}'(t^{1/3}\pi - \lambda t^{-2/3})\mathrm{Ai}'(-\lambda t^{-2/3}) - \mathrm{Bi}'(-\lambda t^{-2/3})\mathrm{Ai}'(t^{1/3}\pi - \lambda t^{-2/3}) = 0.$$

It follows that

$$\mathrm{Ai}'(-\lambda t^{-2/3})\mathrm{Bi}'\Big(t^{1/3}\big(\pi - \frac{\lambda}{t}\big)\Big) - \mathrm{Bi}'(-\lambda t^{-2/3})\mathrm{Ai}'\Big(t^{1/3}\big(\pi - \frac{\lambda}{t}\big)\Big) = 0. \tag{4.1}$$

According to the asymptotic behavior of Airy functions, the equation (4.1) takes the following form as t goes to $+\infty$,

$$\begin{aligned} &\sin\theta(-\lambda t^{-2/3})\exp\Big(\frac{2}{3}\big[t^{1/3}(\pi - \frac{\lambda}{t})\big]^{3/2}\Big) \\ &+ \frac{1}{2}\cos\theta(-\lambda t^{-2/3})\exp\Big(-\frac{2}{3}\big[t^{1/3}(\pi - \frac{\lambda}{t})\big]^{3/2}\Big) = 0. \end{aligned}$$

Then we obtain

$$\tan\theta(-\lambda t^{-2/3}) = -\frac{1}{2}\exp\Big(-\frac{4}{3}\big[t^{1/3}(\pi - \frac{\lambda}{t})\big]^{3/2}\Big).$$

As $\exp\big(-\frac{4}{3}[t^{1/3}(\pi - \frac{\lambda}{t})]^{3/2}\big)$ has a limit 0 when $t \to \infty$. Then

$$\lim_{t\to\infty}\tan\theta(-\lambda t^{-2/3}) = \lim_{t\to\infty}\mathrm{Ai}'(-\lambda t^{-2/3}) = 0,$$

which implies that $\lim_{t\to\infty} -\lambda_n t^{-2/3} = \alpha'_n$. Equivalently

$$\lambda_n = -\alpha'_n t^{2/3} + o(t^{2/3}), \quad \text{as } t \to \infty.$$

Overall, we obtain a lower bound on the fundamental gap

$$\Gamma[q] = \lambda_2 - \lambda_1 = -\alpha'_2 t^{2/3} + o(t^{2/3}) + \alpha'_1 t^{2/3} + o(t^{2/3}) \geq (\alpha'_1 - \alpha'_2)t^{2/3}. \qquad \square$$

Remark 4.3. By Theorem 4.2, we conclude that the fundamental gap is unbounded when t goes to infinity.

Lemma 4.4. *The eigenvalues of Neumann Schrödinger operator with $q(x) = tx$ on $[0,1]$ are given by*

$$\lambda_1 = -\tau_1 t^{2/3} \quad \textit{with } \tau_1 \in (\beta'_2, \beta'_1).$$

Proof. Let

$$\tau_1 = \max\{\tau \in (-\infty, \beta'_1)/D(\tau) = t^{1/3}\}.$$

Assume that $\lambda_n = -\tau_1 t^{2/3}$ with $n \geq 2$, so there exists a solutions $\tilde\tau$ of $f(\tau + \pi t^{1/3}) = f(\tau)$ with $\tilde\tau > \tau$ such that $\lambda_1 = -\tilde\tau t^{2/3}$. Note that by definition of τ_1, $D(\tilde\tau) \neq t^{1/3}$ implies that is $\tilde\tau$ and $\tilde\tau + \pi t^{1/3}$ do not lie in the domains of consecutive branches of f, i.e. if $\tilde\tau \in (\beta'_{n+1}, \beta'_n)$ then $\tilde\tau + \pi t^{1/3} > \beta'_{n-1}$.

By the definition of D, $\tilde\tau$ and $\tilde\tau + D(\tilde\tau)$ lie in the domains of consecutive branches of f that is $\tilde\tau \in (\beta'_{n+1}, \beta'_n)$ implies $\tilde\tau + D(\tilde\tau) \in (\beta'_n, \beta'_{n-1})$. Then $D(\tilde\tau) < t^{1/3}$.

Noting that D is continuous, then there exists $\nu > \tilde\tau$ such that $D(\nu) = t^{1/3}$ so $\nu < \tau_1$. This contradicts $\tilde\tau > \tau_1$. $\square$

We can generalize the previous lemma by the following result.

Lemma 4.5. *The eigenvalues of Neumann Schrödinger operator with $q(x) = tx$ on $[0, 1]$ are given by*

$$\lambda_n = -\tau_n t^{2/3} \quad \text{with } \tau_n \in (\beta'_{n+1}, \beta'_n).$$

Proposition 4.6. *Consider the Neumann Schrödinger problem with linear potential $q(x) = tx$ with $t > 0$ on $[0, 1]$. Let α'_n and β'_n denote the n-th zero of* Ai′ *and* Bi′ *respectively. If $t \geq (\alpha'_1 - \alpha'_2)^3$ then*

$$\Gamma[q] \geq (\alpha'_2 - \beta'_2)t^{2/3}.$$

Proof. We have $D(\alpha'_2) = \alpha'_1 - \alpha'_2$, the fact that D is increasing we obtain

$$\lambda_1 < -\alpha'_2 t^{2/3}.$$

Observing that $\lambda_2 > -\beta'_2 t^{2/3}$. This yields that

$$\Gamma[q] = \lambda_2(q) - \lambda_1(q) \geq \alpha'_2 t^{2/3} - \beta'_2 t^{2/3}.$$

Then we conclude that $\Gamma[q] \geq (\alpha'_2 - \beta'_2)t^{2/3}$. □

Proposition 4.7. *Consider the Neumann Schrödinger problem with linear potential $q(x) = tx$ with $t > 0$ on $[0, \pi]$. Let α'_n and β'_n denote the n-th zero of* Ai′ *and* Bi′ *respectively. If $t \geq \frac{(\alpha'_1 - \alpha'_2)^3}{\pi^3}$ then*

$$\Gamma[q, \pi] \geq (\alpha'_2 - \beta'_2)t^{2/3}.$$

Proof. It is easy to show that

$$\Gamma[q, \pi] = \frac{1}{\pi^2}\Gamma[\pi^2 q(\pi x)].$$

From Proposition 4.6, for $\pi^3 t \geq (\alpha'_1 - \alpha'_2)^3$ we deduce that

$$\Gamma[\pi^2 q(\pi x)] \geq (\alpha'_2 - \beta'_2)\pi^2 t^{2/3}.$$

Consequently, for $t \geq \frac{(\alpha'_1 - \alpha'_2)^3}{\pi^3}$ we obtain $\Gamma[q, \pi] \geq (\alpha'_2 - \beta'_2)t^{2/3}$. □

Example and numerical simulation. Using Mathematica, for

$$\beta'_1 = -2.2944, \quad \beta'_2 = -4.0731, \quad \alpha'_2 = -3.2481, \quad t > 0.1814,$$

we obtain

$q(x) =$	$\frac{1}{2}x$	x	$0,1814x$
$\Gamma \geq$	0.5197	0.825	0.2643

Thus $\Gamma[q_0] \geq 0.2643$ for $q_0(x) = 0.1814x$.

Remark 4.8. Our main results include improvements of the lower bound on the fundamental gap of Robin Schrödinger operators with a convex potential. Meanwhile, when we establish the link between the first eigenvalues and the real roots of the first derivative of the Airy functions Ai′ and Bi′, how small can the fundamental eigenvalue gap be?

5. Appendix: Airy functions [18, 21]

The Airy functions can be defined as the linearly independent solutions to the differential equation

$$\frac{du^2}{dx^2}(x) = xu(x).$$

The first Airy function Ai and the second Airy function Bi have representations as improper Riemann integrals

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\big(\frac{t^3}{3} + xt\big)dt,$$
$$\mathrm{Bi}(x) = \frac{1}{\pi}\int_0^\infty \exp\big(-\frac{t^3}{3} + xt\big)\sin\big(\frac{t^3}{3} + xt\big)dt.$$

The Airy functions Ai and Bi have infinitely many zeros on the negative real axis. We denote those zeros by α_n and β_n, $n \in \mathbb{N}$, respectively, in decreasing order, so that

$$0 > \beta_1 > \alpha_1 > \beta_2 > \alpha_2 > \beta_3 \dots .$$

and let α'_n and β'_n denote the n-th zero of the first derivative of the Airy functions Ai' and Bi' respectively, so that

$$0 > \alpha'_1 > \beta'_1 > \alpha'_2 > \beta'_2 > \alpha'_3 \dots .$$

The asymptotic behavior of the Airy functions Ai and Bi as $t \to \infty$ is given by

$$\mathrm{Ai}(t) \sim \frac{\exp(-\frac{2}{3}t^{3/2})}{2\sqrt{\pi}t^{1/4}}, \quad \mathrm{Bi}(t) \sim \frac{\exp(\frac{2}{3}t^{3/2})}{\sqrt{\pi}t^{1/4}}.$$

And the asymptotic behavior of the Airy functions Ai' and Bi' for $t \to \infty$ is given by

$$\mathrm{Ai}'(t) \sim -\frac{exp(-\frac{2}{3}t^{3/2})}{2\sqrt{\pi}t^{-1/4}}, \quad \mathrm{Bi}'(t) \sim \frac{exp(\frac{2}{3}t^{3/2})}{\sqrt{\pi}t^{-1/4}}.$$

For a negative t,

$$\mathrm{Ai}'(t) = N(t)\sin\theta(t), \quad \mathrm{Bi}'(t) = N(t)\cos\theta(t)$$

where

$$N(t) = \sqrt{(\mathrm{Ai}'(t))^2 + (\mathrm{Bi}'(t))^2}, \quad \theta(t) = \arctan\left(\frac{\mathrm{Ai}'(t)}{\mathrm{Bi}'(t)}\right).$$

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