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DISCRETE ALEKSANDROV SOLUTIONS OF THE MONGE-AMPÈRE EQUATION

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Abstract. We make two relaxations of the Oliker-Prussner method for the Dirichlet problem for the Monge-Ampère equation. First we relax the convexity requirement and consider mesh functions which are only discrete convex. The second relaxation consists in using a finite stencil. The discrete nonlinear equations are solved with a damped Newton's method. We give two proofs of convergence of the resulting scheme for right-hand side a density, on domains which are convex and not necessarily strictly convex, under the assumption that the boundary data has a continuous convex extension. The first proof is based on the notion of Aleksandrov solution while the second uses viscosity solutions.

1. Introduction

In this paper we prove the convergence of a finite difference scheme to weak solutions, in the sense of Aleksandrov and in the sense of viscosity, for the Dirichlet problem for the Monge-Ampère equation

$$\begin{aligned} \det D^2 u = f \quad \text{in } \Omega \\ u = g \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $f \in L^1(\Omega) \cap C(\Omega)$ is a non negative function and Ω is a convex bounded domain of $\mathbb{R}^d$ with boundary $\partial\Omega$. It is assumed that $g \in C(\partial\Omega)$ can be extended to a convex function $\tilde{g} \in C(\overline{\Omega})$. The domain is not assumed to be strictly convex.

Problem (1.1) can be solved through polygonal approximations [28], i.e. with the Oliker-Prussner method [27]. For recent developments on the discretization of (1.1), we refer for example to [12, 23, 24]. The purpose of this paper is to present a technique which can be used to prove convergence of a class of approximations to (1.1) when the domain is convex and not assumed to be strictly convex. As with [22], we consider a method which is medius between the Oliker-Prussner method and finite difference methods. It is relatively simpler to implement than the Oliker-Prussner method, a possible advantage in three dimensions. This is achieved by relaxing the convexity requirement on the approximate solutions. That relaxation leads to a wide stencil scheme, which we further relax by using a finite stencil. The discrete nonlinear equations are solved with a damped Newton's method. Convergence of

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the method is first given in the setting of Aleksandrov solutions, using an equicontinuity argument and a recent result [2] stating conditions under which the uniform limit of discrete solutions satisfies the boundary condition strongly. The scheme we analyze, leads to a set function that overestimates the discrete Monge-Ampère measure defined through a discrete version of the subdifferential. This allows us to use essentially the same tools as in the Aleksandrov theory of (1.1), c.f. [2].

Under the above assumptions, Aleksandrov solutions are equivalent to viscosity solutions. The hallmark of the Barles-Souganidis approach for convergence to viscosity solutions, is that no equicontinuity is used. Ingredients are stability, consistency and monotonicity of the scheme. As well as a comparison principle for Dirichlet boundary conditions in the sense of viscosity, which is not available for the Monge-Ampère equation [10, 13, 18, 23, 25]. Here, using the equicontinuity of the discrete solutions, and under the assumption that our scheme is (pointwise) consistent, we obtain a uniform limit of discrete solutions which is shown to be a viscosity solution of the equation satisfying the boundary condition strongly, hence is unique by the comparison principle for boundary conditions imposed strongly. This argument requires f to be integrable and is applicable to some other discretizations.

We also give some convergence results in the case where the right hand side is a sum of Dirac masses instead of a positive density. However, in that case the right-hand side becomes singular unlike in the case of the Oliker-Prussner method, making the use of a damped Newton's method for the relaxed scheme not feasible. Nethertheless, these results could be useful for the analysis of schemes such as the one in [7].

This article is organized as follows. In the next section we collect some notation used throughout the paper, present and study the numerical scheme. Convergence results are given in section 3. We finish with some numerical experiments.

2. Preliminaries

We use the notation $\|\cdot\|$ for the Euclidean norm of $\mathbb{R}^d$. Let h be a small positive parameter and let

$$\mathbb{Z}^d_h = \{mh : m \in \mathbb{Z}^d\},$$

denote the orthogonal lattice with mesh length h. Let also $(r_1, \dots, r_d)$ denote the canonical basis of $\mathbb{R}^d$. We define

$$\Omega_h = \Omega \cap \mathbb{Z}^d_h.$$

For a function $u \in C(\Omega)$ its restriction on Ω_h is also denoted u by an abuse of notation. For $x \in \Omega_h$ and $e \in \mathbb{Z}^d$ let

$$h^e_x = \sup\{rh : r \in [0,1] \text{ and } x + rhe \in \overline{\Omega}\}.$$

Next, let $V \subset \mathbb{Z}^d \setminus \{0\}$ such that $\{r_1, \dots, r_d\} \subset V$ and such that for $e \in V$, $-e \in V$. We define

$$\partial\Omega_h = \{x \in \partial\Omega : \exists y \in \Omega_h \text{ and } e \in V \text{ such that } x = y + h^e_y e\}, \tag{2.1}$$

and denote by $\mathcal{U}_h$ the linear space of mesh functions, i.e. real-valued functions defined on

$$\mathcal{N}_h := \Omega_h \cup \partial\Omega_h.$$

For $x \in \Omega_h$, $e \in \mathbb{Z}^d, e \neq 0$ such that $x \pm he \in \mathcal{N}_h$ and $u_h \in \mathcal{U}_h$, let

$$\Delta_e u_h(x) = \frac{2}{h_x^e + h_x^{-e}}\Big(\frac{u_h(x + h_x^e e) - u_h(x)}{h_x^e} + \frac{u_h(x - h_x^{-e} e) - u_h(x)}{h_x^{-e}}\Big).$$

Definition 2.1. We say that a mesh function v_h is *discrete convex* if $\Delta_e v_h(x) \geq 0$ for all $x \in \Omega_h$ and $e \in V \subset \mathbb{Z}^d$.

We denote by $\mathcal{C}_h$ the cone of discrete convex mesh functions. The restriction of a convex function to Ω_h is a discrete convex mesh function.

2.1. **Aleksandrov solutions.** The material in this subsection is taken from [15] to which we refer for proofs. Let Ω be an open subset of $\mathbb{R}^d$ and let us denote by $\mathcal{P}(\mathbb{R}^d)$ the set of subsets of $\mathbb{R}^d$.

Definition 2.2. Let $u : \Omega \to \mathbb{R}$. The normal mapping of u, or subdifferential of u is the set-valued mapping $\partial u : \Omega \to \mathcal{P}(\mathbb{R}^d)$ defined by

$$\partial u(x_0) = \{p \in \mathbb{R}^d : u(x) \geq u(x_0) + p \cdot (x - x_0), \text{ for all } x \in \Omega\}. \tag{2.2}$$

Let $|E|$ denote the Lebesgue measure of the measurable subset $E \subset \Omega$. For $E \subset \Omega$, we define

$$\partial u(E) = \cup_{x \in E} \partial u(x).$$

Theorem 2.3 ([15, Theorem 1.1.13])**.** *If u is continuous on Ω, the class*

$$\mathcal{S} = \{E \subset \Omega : \partial u(E) \text{ is Lebesgue measurable}\},$$

is a Borel σ-algebra and the set function $M[u] : \mathcal{S} \to \overline{\mathbb{R}}$ defined by

$$M[u](E) = |\partial u(E)|,$$

is a measure, finite on compact subsets, called the Monge-Ampère measure associated with the function u.

We can now define the notion of Aleksandrov solution of the Monge-Ampère equation.

Definition 2.4. Let $\Omega \subset \mathbb{R}^d$ be open and convex. Given a Borel measure ν on Ω, a convex function $u \in C(\Omega)$ is an Aleksandrov solution of

$$\det D^2 u = \nu,$$

if the associated Monge-Ampère measure $M[u]$ is equal to ν.

We recall an existence and uniqueness result for the solution of (1.1).

Proposition 2.5 ([16, Theorem 1.1])**.** *Let Ω be a bounded convex domain of $\mathbb{R}^d$. Assume ν is a finite Borel measure and $g \in C(\partial\Omega)$ can be extended to a function $\tilde{g} \in C(\overline{\Omega})$ which is convex in Ω. Then the Monge-Ampère equation* (1.1) *has a unique convex Aleksandrov solution in $C(\overline{\Omega})$.*

Definition 2.6. A sequence μ_n of Borel measures converges to a Borel measure μ if $\mu_n(B) \to \mu(B)$ for any Borel set B with $\mu(\partial B) = 0$.

We note that there are several equivalent definitions of weak convergence of measures which can be found for example in [11, Theorem 1, section 1.9].

2.2. **Discretizations of the normal mapping.** For a mesh function $u_h \in \mathcal{C}_h$, the discrete normal mapping of u_h at the point $x \in \Omega \cap \mathbb{Z}^d_h$ is defined as

$$\partial_h u_h(x) = \{p \in \mathbb{R}^d : u_h(x + h^e_x e) \geq u_h(x) + p \cdot (h^e_x e) \, \forall e \in \mathbb{Z}^d\}.$$

For a subset $E \subset \Omega$, we define

$$\partial_h u_h(E) = \cup_{x \in E \cap \mathbb{Z}^d_h} \partial_h u_h(x),$$

which is Borel measurable for E Borel measurable. The proof is essentially the same as the corresponding one at the continuous level [4, p. 117–118]. Put

$$M_h[u_h](E) = |\partial_h u_h(E)| \quad \text{for a Borel set } E.$$

Note that if $E \cap \mathbb{Z}^d_h = \{x\}$, we have $M_h[u_h](E) = M_h[u_h](\{x\})$. We will make the abuse of notation

$$M_h[u_h](x) = M_h[u_h](\{x\}).$$

A numerical scheme based on $M_h[u_h]$ would require a wide scheme. We will use $M_h[u_h]$ in our proof of convergence. We now consider a discrete Monge-Ampère measure based on the finite stencil V. Put

$$\partial_V u_h(x) = \{p \in \mathbb{R}^d, u_h(x + h^e_x e) \geq u_h(x) + p \cdot (h^e_x e) \, \forall e \in V\},$$

and $\partial_V u_h(E) = \cup_{x \in E \cap \mathbb{Z}^d_h} \partial_V u_h(x)$ with

$$M_V[u_h](E) = |\partial_V u_h(E)| \text{ for a Borel set } E.$$

We have

$$M_V[u_h](x) \geq M_h[u_h](x), \quad \forall x \in \Omega_h, \tag{2.3}$$

since $\partial_h u_h(x) \subset \partial_V u_h(x)$.

2.3. **Viscosity solutions of the elliptic Monge-Ampère equation.** A convex function $u \in C(\overline{\Omega})$ is a viscosity solution of (1.1) if $u = g$ on $\partial\Omega$ and for all $\phi \in C^2(\Omega)$ the following holds

- at each local maximum point x_0 of $u - \phi$, $f(x_0) \leq \det D^2\phi(x_0)$
- at each local minimum point x_0 of $u-\phi$, $f(x_0) \geq \det D^2\phi(x_0)$, if $D^2\phi(x_0) \geq 0$, i.e. $D^2\phi(x_0)$ has positive eigenvalues.

As explained in [17], the requirement $D^2\phi(x_0) \geq 0$ in the second condition above is natural for the two dimensional case we consider. The space of test functions in the definition above can be restricted to the space of strictly convex quadratic polynomials [15, Remark 1.3.3].

An upper semi-continuous convex function u is said to be a viscosity sub solution of $\det D^2u(x) = f(x)$ if the first condition holds and a lower semi-continuous convex function is said to be a viscosity super solution when the second holds. A viscosity solution of (1.1) is a continuous function which satisfies the boundary condition and is both a viscosity sub solution and a viscosity super solution. Note that the notion of viscosity solution is a pointwise notion, i.e. conditions will be checked at a point in the domain.

For further reference, we recall the comparison principle of sub and super solutions, [17, Theorem V. 2].

Theorem 2.7. *Let u and v be respectively sub and super solutions of* $\det D^2u(x) = f(x)$ *in* Ω. *Then if* $\sup_{x\in\partial\Omega} \max(u(x) - v(x), 0) = M$, *then* $u(x) - v(x) \leq M$ *in* Ω.

2.3.1. *Equivalence with Aleksandrov solutions.* For $f > 0$, a convex function $u \in C(\overline{\Omega})$ is an Aleksandrov solution of (1.1) if and only if it is a viscosity solution of (1.1), [15, Propositions 1.3.4 and 1.7.1]. The equivalence of viscosity and Aleksandrov solutions in the degenerate case $f \geq 0$ is discussed in [3].

2.4. **The numerical scheme.** We consider the following discretization of (1.1): find $u_h \in \mathcal{C}_h$ such that

$$\begin{aligned} M_V[u_h](x) &= h^d f(x), \quad x \in \Omega_h \\ u_h(x) &= g(x), \quad x \in \partial\Omega_h. \end{aligned} \tag{2.4}$$

We establish the stability, unicity and existence of solutions to (2.4). We first recall the Brunn-Minkowski's inequality [29].

Lemma 2.8. *For two nonempty, compact convex sets K and L, their Minkowski sum is defined as*

$$K + L = \{a + b, a \in K \text{ and } b \in L\}.$$

We have

$$|K + L|^{1/d} \geq |K|^{1/d} + |L|^{1/d}. \tag{2.5}$$

Lemma 2.9. *Given $x \in \Omega_h$ the operator $v_h \to (M_V[v_h](x))^{1/d}$ is concave on $\mathcal{C}_h$.*

Proof. We recall that given a set K and $\lambda \in \mathbb{R}$, $\lambda K = \{\lambda x, x \in K\}$. We observe that for $\lambda > 0$, $p \in \partial_V v_h(x)$ if and only if $\lambda p \in \partial_V(\lambda v_h)(x)$. Thus by the positive homogeneity (of degree d) of volume in $\mathbb{R}^d$

$$(M_V[\lambda v_h](x))^{1/d} = \lambda(M_V[v_h](x))^{1/d}.$$

It is therefore enough to prove that for $v_h, w_h \in \mathcal{C}_h$, we have

$$(M_V[v_h + w_h](x))^{1/d} \geq (M_V[v_h](x))^{1/d} + (M_V[w_h](x))^{1/d}. \tag{2.6}$$

Next, we note that

$$\partial_V v_h(x) + \partial_V w_h(x) \subset \partial_V(v_h + w_h)(x),$$

and thus $|\partial_V(v_h + w_h)(x)| \geq |\partial_V v_h(x) + \partial_V w_h(x)|$. We may assume that $\partial_V v_h(x)$ and $\partial_V w_h(x)$ are nonempty. Assuming that $\partial_V v_h(x)$ is compact and convex, (2.6) follows from (2.5).

Using the definition and the canonical basis of $\mathbb{R}^d$ one shows that $\partial_V v_h(x)$ is bounded. Thus $\partial_V v_h(x)$ is compact since it can be shown to be a closed set. The convexity of $\partial_V v_h(x)$ is a consequence of its definition. This concludes the proof. □

Lemma 2.10. *Let $C_y(x) = \|y - x\|$ denote the cone with vertex $y \in \Omega_h$. Then*

$$M_h[C_y](y) \geq \omega_d > 0,$$

where ω_d is the volume of the closed unit ball.

Proof. We have $C_y(y) = 0$ and $p \in \partial_V C_y(y)$ if and only if $p \cdot e \geq -\|e\| \forall e \in V$. Clearly $\partial_V C_y(y)$ contains the closed unit ball with volume ω_d. This concludes the proof. □

2.4.1. *Stability.* Since $f \in L^1(\Omega) \cap C(\Omega)$ and $f \geq 0$, we have

$$\sum_{x \in \Omega_h} h^d f(x) \leq A, \tag{2.7}$$

with A a number independent of h, for h sufficiently small. For $x \in \Omega$ we denote by $d(x, \partial\Omega)$ the distance of x to $\partial\Omega$. For a subset S of Ω, $\operatorname{diam}(S)$ denotes its diameter.

Lemma 2.11. *Let $v_h \in \mathcal{C}_h$. Then*

$$\max_{x \in \mathcal{N}_h} v_h(x) \leq \max_{x \in \partial\Omega_h} v_h(x).$$

Proof. Suppose there is $x_0 \in \Omega_h$ such that $\max_{x \in \mathcal{N}_h} v_h(x) = v_h(x_0)$ and $v_h(x_0) > \max_{x \in \partial\Omega_h} v_h(x)$.

For all $e \in V$, we have $v_h(x_0) \geq v_h(x_0 + h^e_{x_0} e)$ and $v_h(x_0) \geq v_h(x_0 - h^{-e}_{x_0} e)$. This implies that $\Delta_e v_h(x_0) \leq 0$ and hence $\Delta_e v_h(x_0) = 0$ since v_h is discrete convex. We have

$$\frac{v_h(x_0 + h^e_{x_0} e) - v_h(x_0)}{h^e_{x_0}} = \frac{v_h(x_0) - v_h(x_0 - h^{-e}_{x_0} e)}{h^{-e}_{x_0}}.$$

Since the left hand side of the above equation is non positive and the right hand side non negative, we conclude that $v_h(x_0) = v_h(x_0 + h^e_{x_0} e) = v_h(x_0 - h^{-e}_{x_0} e)$, i.e. the maximum is also reached at $x_0 + h^e_{x_0} e$ and $x_0 - h^{-e}_{x_0} e$. Repeating this argument, we may assume that the maximum is reached at an interior point x_0 such that $x_0 + h^e_{x_0} e \in \partial\Omega_h$. Since by assumption $v_h(x_0) > v_h(x_0 + h^e_{x_0} e)$, we obtain $\Delta_e v_h(x_0) < 0$, contradicting the assumption $v_h \in \mathcal{C}_h$. ☐

Lemma 2.12 below is an analogue of [15, Lemma 1.4.1], c.f. [26], and is a discrete version of the Aleksandrov-Bakelman-Pucci's maximum principle [30, Theorem 8.1], an analogue of which can be found in [19].

Lemma 2.12. *Let $u_h \in \mathcal{C}_h$ such that $u_h \geq 0$ on $\partial\Omega_h$. Then for $x \in \Omega_h$*

$$u_h(x) \geq -C(d)\big[\operatorname{diam}(\Omega)^{d-1} d(x, \partial\Omega) M_h[u_h](\Omega_h)\big]^{1/d},$$

for a positive constant $C(d)$ which depends only on d.

Theorem 2.13. *Solutions u_h to (2.4) are uniformly bounded.*

Proof. By Lemma 2.11, we have

$$u_h(x) \leq \max_{x \in \partial\Omega_h} g(x). \tag{2.8}$$

By Lemma 2.12,

$$u_h(x) - \min_{x \in \partial\Omega_h} g(x) \geq -C(d)\big[\operatorname{diam}(\Omega)^{d-1} d(x, \partial\Omega) M_h[u_h](\Omega_h)\big]^{1/d}.$$

Since u_h solves (2.4), by (2.7) and (2.3)

$$A \geq \sum_{x \in \Omega_h} h^d f(x) = \sum_{x \in \Omega_h} M_V[u_h](x) \geq \sum_{x \in \Omega_h} M_h[u_h](x) \geq M_h[u_h](\Omega_h).$$

In addition $d(x, \partial\Omega) \leq \operatorname{diam}(\Omega)$. We conclude that $u_h(x) \geq \min_{x \in \partial\Omega_h} g(x) - C$, for a constant C. Combined with (2.8), we have shown that solutions u_h to (2.4) are uniformly bounded. ☐

2.4.2. *Uniqueness.*

Theorem 2.14. *Under the assumption that $f > 0$ on Ω_h, Problem (2.4) has a unique solution u_h.*

Proof. We consider the convex envelope of the mesh function u_h

$$\Gamma(u_h)(x) = \sup_{L \text{affine}} \{L(x) : L(y) \leq u_h(y) \text{ for all } y \in \mathcal{N}_h\},$$

which is a piecewise linear convex function, c.f. for example [2, p. 11]. We note that $\Gamma(u_h)$ depends on the stencil V. This notion of convex envelope generalizes the one used in [2] where we used $V = \mathbb{Z}^d \setminus \{0\}$. The following result is an analogue of [2, Lemmas 6 and 7] and [2, Theorem 4] where we considered $\partial_h u_h$. The proofs are identical.

If $x \in \Omega_h$ and $\Gamma(u_h)(x) \neq u_h(x)$, then $\partial_V u_h(x) = \emptyset$. Moreover, for a subset $E \subset (\text{conv}(\mathcal{N}_h))^\circ$, $\partial_V u_h(E) = \partial\Gamma(u_h)(E)$ up to a set of measure 0 and thus

$$M_V[u_h](E) = M[\Gamma(u_h)](E).$$

Since g extends to a continuous convex function on $\overline{\Omega}$, analogous to [2, Lemma 5], $\Gamma(u_h) = u_h$ on $\partial\Omega_h$ if u_h solves (2.4).

Next, under the assumption that $f > 0$ on Ω_h, $\partial_V u_h(x) \neq \emptyset$ for all $x \in \Omega_h$, and we conclude that if u_h solves (2.4), $\Gamma(u_h)$ solves the Monge-Ampère equation

$$M[\Gamma(u_h)](E) = \sum_{x \in E \cap \Omega_h} h^d f(x),$$

for each Borel set $E \subset (\text{conv}(\mathcal{N}_h))^\circ$ with $\Gamma(u_h) = g$ on $\partial\Omega_h$ and hence $\Gamma(u_h)$ is a prescribed piecewise linear convex function on the boundary of $\text{conv}(\mathcal{N}_h)$. By Proposition 2.5, the solution is unique, and since $\Gamma(u_h) = u_h$ on $\partial\Omega_h$, the solution u_h is unique. □

2.4.3. *Existence.* We show that minimizers of a convex functional over a convex set solve (2.4). For $v_h \in \mathcal{U}_h$ and $i = 1, \ldots, d$ we consider the first-order difference like operator defined by

$$\partial_-^i v_h(x) := \frac{v_h(x) - v_h(x - h_x^{-r_i} r_i)}{\sqrt{h_x^{-r_i}}}, x \in \Omega_h,$$

and the convex functional

$$J_h(v_h) = \sum_{x \in \Omega_h} \|D_h v_h(x)\|^2,$$

where $D_h v_h$ is given by

$$D_h v_h(x) = (\partial_-^i v_h(x))_{i=1,\ldots,d}.$$

We seek a minimizer of J_h over

$$S_h = \{v_h \in \mathcal{C}_h : v_h = g_h \text{ on } \partial\Omega_h, \text{ and } (M_V[v_h](x))^{1/d} \geq f(x)^{1/d}, x \in \Omega_h\}. \quad (2.9)$$

The set S_h is a discrete version of

$$S = \{v \in C(\overline{\Omega}) : v \text{ convex }, v = g \text{ on } \partial\Omega, \text{ and } \det D^2 v \geq f\}.$$

It is known, [21], that the solution u of (1.1) is the maximal element of S. This observation may be used to prove the existence of a discrete solution. The proof below is motivated by a variational characterization of solutions of (1.1), [21, Section 4], and an observation in [20, p. 86] on discrete Monge-Ampère equations.

Lemma 2.15. *The set S_h is convex and nonempty.*

Proof. The convexity of S_h follows from Lemma 2.9. For each y in Ω_h, let q_y be a cone such that $M_V[q_y](y) \geq f(y)$. For example, we may define q_y by

$$q_y(x) = \Big(\frac{f(y)}{\omega_d}\Big)^{1/d} C_y(x).$$

Put $\hat{q} = \sum_{y \in \Omega_h} q_y$. Since g is bounded on $\partial\Omega$, we can find a number κ such that $\hat{q} - \kappa \leq g$ on $\partial\Omega$. We define $w_h \in \mathcal{U}_h$ by

$$\begin{gathered} w_h(x) = \hat{q}(x) - \kappa, x \in \Omega_h \\ w_h = g \text{ on } \partial\Omega_h. \end{gathered}$$

We claim that $w_h \in S_h$.

Let $x \in \Omega_h$ and $e \in V$. Either $w_h(x + h_x^e e) = \hat{q}(x + h_x^e e) - \kappa$ or $w_h(x + h_x^e e) = g(x + h_x^e e) \geq \hat{q}(x + h_x^e e) - \kappa$. Similarly $w_h(x - h_x^{-e} e) \geq \hat{q}(x - h_x^{-e} e) - \kappa$. We conclude using the convexity of $\hat{q}$ that

$$\begin{aligned} \frac{h_x^e + h_x^{-e}}{2} \Delta_e w_h(x) &\geq \frac{\hat{q}(x + h_x^e e) - \kappa - w_h(x)}{h_x^e} + \frac{\hat{q}(x - h_x^{-e} e) - \kappa - w_h(x)}{h_x^{-e}} \\ &= \frac{h_x^e + h_x^{-e}}{2} \Delta_e \hat{q}(x) \geq 0. \end{aligned}$$

Thus $w_h \in \mathcal{C}_h$. Next, we prove that $\partial_V \hat{q}(x) \subset \partial_V w_h(x)$ for $x \in \Omega_h$.

Since $w_h = \hat{q}$ up to a constant on Ω_h, we only need to check that for $p \in \partial_V \hat{q}(x)$ we have $p \cdot (h_x^e e) \leq w_h(x + h_x^e e) - w_h(x)$ when $x + h_x^e e \in \partial\Omega_h$. Let thus $p \in \partial_V \hat{q}(x)$ such that $x + h_x^e e \in \partial\Omega_h$. We have

$$\begin{aligned} p \cdot e &\leq \hat{q}(x + h_x^e e) - \hat{q}(x) = \hat{q}(x + h_x^e e) - \kappa - w_h(x) \\ &\leq g(x + h_x^e e) - w_h(x) \qquad = w_h(x + h_x^e e) - w_h(x). \end{aligned}$$

We conclude that $M_V[w_h](x) \geq M_V[\hat{q}](x)$. Therefore by the Brunn-Minkowski inequality (2.5),

$$M_V[w_h](x)^{1/d} \geq \sum_{y \in \Omega_h} M_V[q_y](x)^{1/d} \geq f(x)^{1/d}.$$

This concludes the proof. □

We observe that if $p \in \partial_V u_h(x)$, $x \in \Omega_h$, we have

$$\frac{u_h(x) - u_h(x - h_x^{-e} e)}{h_x^{-e}} \leq p \cdot e \leq \frac{u_h(x + h_x^e e) - u_h(x)}{h_x^e}. \tag{2.10}$$

Thus

$$M_V[u_h](x) \leq \prod_{i=1}^d \frac{h_x^e + h_x^{-e}}{2} \Delta_{r_i} u_h(x). \tag{2.11}$$

We define a discrete norm on $\mathcal{U}_h$ by

$$\|v_h\|_{0,h}^2 = h^d \sum_{x \in \Omega_h} v_h(x)^2,$$

and a semi norm by

$$|v_h|_{1,h}^2 = h^d \sum_{x \in \Omega_h} \sum_{i=1}^d \frac{1}{h_x^{-r_i}} \big(\partial_-^i v_h(x)\big)^2,$$

an analogue of a Sobolev semi-norm.

Lemma 2.16. *We have an analogue of Poincaré's inequality,*

$$C\|v_h\|_{0,h} \leq |v_h|_{1,h}, \text{ for } v_h = 0 \text{ on } \partial\Omega_h, \tag{2.12}$$

for a constant $C > 0$ independent of h.

Proof. Given $x \in \Omega_h$ and $1 \leq i \leq d$, let $m_x^i = \max\{m \in \mathbb{N} : x - mh_x^{-r_i} r_i \in \Omega_h\}$, and put $y_x^i = x - m_x^i h r_i$. Thus $y_x^i - h_{y_x^i}^{-r_i} r_i \in \partial\Omega_h$ and by assumption $v_h(y_x^i - h_{y_x^i}^{-r_i} r_i) = 0$. We have

$$\begin{aligned} v_h(x) &= \sum_{j=0}^{m_x^i - 1} v_h(x - jhr_i) - v_h(x - (j-1)hr_i) + v_h(y_x^i) \\ &= \sum_{j=0}^{m_x^i - 1} h \frac{v_h(x - jhr_i) - v_h(x - (j+1)hr_i)}{h} + h_{y_x^i}^{-r_i} \frac{v_h(y_x^i) - v_h(y_x^i - h_{y_x^i}^{-r_i} r_i)}{h_{y_x^i}^{-r_i}}. \end{aligned}$$

It follows that

$$\begin{aligned} v_h(x)^2 &\leq \Big(\sum_{j=0}^{m_x^i - 1} h^2 + (h_{y_x^i}^{-r_i})^2 \Big) \Big(\sum_{j=0}^{m_x^i - 1} \frac{(v_h(x - jhr_i) - v_h(x - (j+1)hr_i))^2}{h^2} \\ &\quad + \frac{(v_h(y_x^i) - v_h(y_x^i - h_{y_x^i}^{-r_i} r_i))^2}{(h_{y_x^i}^{-r_i})^2} \Big) \\ &\leq m_x^i h^2 \Big(\sum_{j=0}^{m_x^i - 1} \frac{1}{h} (\partial_-^i v_h(x - jhr_i))^2 + \frac{1}{h_{y_x^i}^{-r_i}} (\partial_-^i v_h(y_x^i))^2 \Big). \end{aligned}$$

Since $m_x^i h$ and h are bounded by the diameter of Ω, for some constant $C > 0$ independent of h,

$$\begin{aligned} \sum_{x \in \Omega_h} v_h(x)^2 &\leq \frac{1}{C} \sum_{x \in \Omega_h} \sum_{j=0}^{m_x^i - 1} \frac{1}{h} (\partial_-^i v_h(x - jhr_i))^2 + \frac{1}{h_{y_x^i}^{-r_i}} (\partial_-^i v_h(y_x^i))^2 \\ &\leq \frac{1}{C} \sum_{x \in \Omega_h} \sum_{i=1}^{d} \frac{1}{h_x^{-r_i}} (\partial_-^i v_h(x))^2, \end{aligned}$$

which concludes the proof. □

Lemma 2.17. *The functional J_h is coercive on S_h, i.e.*

$$J_h(v_h) \to \infty \quad \text{as } \|v_h\|_{0,h} \to \infty, \; v_h \in S_h.$$

Proof. We will assume that h is fixed. We first note that for all $x \in \Omega_h$ and $e \in V$, $h_x^e > 0$. Let $\alpha_h = \min\{h_x^e, x \in \Omega_h, e \in V\}$. Thus for $i = 1, \dots, d$, $1/h_x^{-r_i} \geq \alpha_h/(h_x^{-r_i})^2$ and $h^d J_h(v_h) \geq \alpha_h |v_h|_{1,h}^2$.

Let $\Phi(p) = \|p\|^2$. Using Φ' to denote the Fréchet derivative of Φ, we have

$$(\Phi'(p) - \Phi'(q))(p - q) = 2\|p - q\|^2.$$

We argue as in [31, p. 550] and put $\phi(t) = \Phi(tD_h v_h(x) + (1-t)D_h w_h(x))$. Then

$$\Phi(D_h v_h(x)) - \Phi(D_h w_h(x))$$

$$
\begin{aligned}
&= \phi(1) - \phi(0) = \int_0^1 \phi'(t) dt \\
&= \int_0^1 \Phi'(tD_h v_h(x) + (1-t) D_h w_h(x))(D_h v_h(x) - D_h w_h(x)) dt \\
&= \int_0^1 \Big(\Phi'(tD_h v_h(x) + (1-t) D_h w_h(x)) - \Phi'(D_h w_h(x)) \Big)(D_h v_h(x) - D_h w_h(x)) \\
&\quad + \Phi'(D_h w_h(x))(D_h v_h(x) - D_h w_h(x)) dt \\
&\geq \|D_h v_h(x) - D_h w_h(x)\|^2 - 2\|D_h w_h(x)\| \, \|D_h v_h(x) - D_h w_h(x)\|.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
&h^d J_h(v_h) - h^d J_h(w_h) \\
&= h^d \sum_{x \in \Omega_h} \Phi(D_h v_h(x)) - \Phi(D_h w_h(x)) \\
&\geq h^d \sum_{x \in \Omega_h} \|D_h v_h(x) - D_h w_h(x)\|^2 - 2h^d \sum_{x \in \Omega_h} \|D_h w_h(x)\| \, \|D_h v_h(x) - D_h w_h(x)\|.
\end{aligned}
$$

Next

$$
\begin{aligned}
&\sum_{x \in \Omega_h} \|D_h w_h(x)\| \, \|D_h v_h(x) - D_h w_h(x)\| \\
&\leq \Big(\sum_{x \in \Omega_h} \|D_h w_h(x)\|^2 \Big)^{1/2} \Big(\sum_{x \in \Omega_h} \|D_h v_h(x) - D_h w_h(x)\|^2 \Big)^{1/2} \\
&= J_h(w_h)^{1/2} J_h(v_h - w_h)^{1/2}.
\end{aligned}
$$

Next, if $z_h = 0$ on $\partial\Omega_h$, we claim that $J_h(z_h) \leq (4d/\alpha_h) h^{-d} \|z_h\|_{0,h}^2$. Indeed

$$
\begin{aligned}
J_h(z_h) &= \sum_{x \in \Omega_h} \|D_h z_h(x)\|^2 = \sum_{x \in \Omega_h} \sum_{i=1}^d (\partial_-^i z_h(x))^2 \\
&= \sum_{x \in \Omega_h} \sum_{i=1}^d \frac{(z_h(x) - z_h(x - h_x^{-r_i} r_i))^2}{h_x^{-r_i}} \\
&\leq \frac{2}{\alpha_h} \sum_{x \in \Omega_h} \sum_{i=1}^d z_h(x)^2 + z_h(x - h_x^{-r_i} r_i)^2 \\
&\leq \frac{2d}{\alpha_h} \sum_{x \in \Omega_h} z_h(x)^2 + \frac{2}{\alpha_h} \sum_{i=1}^d \sum_{x \in \Omega_h} z_h(x - h_x^{-r_i} r_i)^2,
\end{aligned}
$$

which using $z_h = 0$ on $\partial\Omega_h$ gives

$$
J_h(z_h) \leq \frac{2d}{\alpha_h} \sum_{x \in \Omega_h} z_h(x)^2 + \frac{2}{\alpha_h} \sum_{i=1}^d \sum_{x \in \Omega_h} z_h(x)^2 \leq \frac{4d}{\alpha_h} \sum_{x \in \Omega_h} z_h(x)^2 = \frac{4d}{\alpha_h} h^{-d} \|z_h\|_{0,h}^2.
$$

If $v_h - w_h = 0$ on $\partial\Omega_h$, we obtain $J_h(v_h - w_h) \leq (4d/\alpha_h) h^{-d} \|v_h - w_h\|_{0,h}^2$. Therefore, for v_h and $w_h \in S_h$

$$
h^d J_h(v_h) - h^d J_h(w_h) \geq \alpha_h |v_h - w_h|_{1,h}^2 - 2\Big(\frac{4d}{\alpha_h}\Big)^{1/2} J_h(w_h)^{1/2} \|v_h - w_h\|_{0,h},
$$

and so using (2.12),

$$h^d J_h(v_h) - h^d J_h(w_h) \geq C\alpha_h \|v_h - w_h\|_{0,h}^2 - 2\Big(\frac{4d}{\alpha_h}\Big)^{1/2} J_h(w_h)^{1/2} \|v_h - w_h\|_{0,h},$$

from which the coercivity of J_h on S_h holds. □

Theorem 2.18. *The functional J_h has a minimizer u_h in S_h and u_h solves the finite difference equations* (2.4).

Proof. Since J_h is convex and coercive on S_h and S_h is nonempty, closed and convex, it follows that the functional J_h has a minimizer u_h on S_h.

We now show that u_h solves the finite difference equations (2.4). To this end, it suffices to show that

$$M_V[u_h] = h^d f \text{ on } \Omega_h.$$

Let us assume to the contrary that there exists $x_0 \in \Omega_h$ such that

$$M_V[u_h](x_0) > h^d f(x_0) \geq 0. \tag{2.13}$$

By (2.10), if there were a direction $e \in V$ such that $\Delta_e u_h(x_0) = 0$, we would have $\partial_V u_h(x_0)$ contained in the hyperplane $p \cdot e = (u_h(x_0 + h_x^e e) - u_h(x_0))/h_x^e = (u_h(x_0) - u_h(x_0 - h_x^{-e} e))/h_x^{-e}$, and hence $M_V[u_h](x_0) = 0$ contradicting (2.13). We conclude that for all $e \in V$ $\Delta_e u_h(x_0) > 0$. Let

$$\epsilon_0 = \inf\{\Delta_e u_h(x_0) : e \in V\}.$$

We recall that $M_V[u_h](x)$ is the volume of a polygon since it is the volume of a domain obtained as an intersection of half-spaces $p \cdot (h_x^e e) \leq u_h(x + h_x^e e) - u_h(x)$. Moreover $\partial_V u_h(x)$ is bounded by (2.11). The vertices of the polygon have coordinates linear combinations of the values $u_h(y), y \in \mathcal{N}_h$. It is known that the volume of a polygon is a polynomial function, hence a continuous function, of the coordinates of its vertices [1]. Thus the mapping $E : \mathbb{R} \to \mathbb{R}$ which maps the value of a mesh function v_h at x_0 to $M_V[u_h](x_0)$ is finite valued and continuous. By (2.13), with $r_0 = u_h(x_0)$, $E(r_0) > h^d f(x_0)$. Therefore there exists $\epsilon_1 > 0$ such that for $|r - r_0| < \epsilon_1$, we have $E(r) > h^d f(x_0)$. Finally, put $\epsilon = h_{x_0}^e h_{x_0}^{-e} \min(\epsilon_0, \epsilon_1)$. We define w_h by

$$w_h(x) = u_h(x), \quad x \neq x_0, \quad w_h(x_0) = u_h(x_0) + \frac{\epsilon}{4}.$$

By construction $w_h = g_h$ on$\partial\Omega_h$. For $x \neq x_0$, we have

$$\Delta_e w_h(x) = \Delta_e u_h(x), \Delta_e w_h(x) = \Delta_e u_h(x) + \frac{\epsilon}{4} \frac{2}{h_x^e + h_x^{-e}} \frac{1}{h_x^e}$$

or

$$\Delta_e w_h(x) = \Delta_e u_h(x) + \frac{\epsilon}{4} \frac{2}{h_x^e + h_x^{-e}} \frac{1}{h_x^{-e}}.$$

Moreover $\Delta_e w_h(x_0) = \Delta_e u_h(x_0) - \epsilon/(2h_{x_0}^e h_{x_0}^{-e}) \geq \epsilon_0 - \epsilon/(2h_{x_0}^e h_{x_0}^{-e}) \geq \epsilon/(2h_{x_0}^e h_{x_0}^{-e}) > 0$ by the definition of ϵ. We conclude that $w_h \in \mathcal{C}_h$.

Also by construction, $M_V[w_h](x_0) = E(r_0 + \epsilon/4) > h^d f(x_0)$. We claim that for $x \neq x_0$ $M_V[w_h](x) \geq M_V[u_h](x)$. Let $p \in \mathbb{R}^d$ and $e \in V$ such that $p \cdot (h_x^e e) \leq u_h(x + h_x^e e) - u_h(x)$. Either $u_h(x + h_x^e e) = w_h(x + h_x^e e)$ or $u_h(x + h_x^e e) = w_h(x + h_x^e e) - \epsilon/4$. This gives $p \cdot (h_x^e e) \leq w_h(x + h_x^e e) - w_h(x)$. This proves the claim. We conclude that $M_V[w_h](x) \geq h^d f(x)$ for all $x \in \Omega_h$.

It remains to show that $J_h(w_h) < J_h(u_h)$. Let Ω_{x_0} denote the subset of Ω_h consisting in x_0 and the points $x_0 + h_{x_0}^{r_j} r_j$, $j = 1, \dots, d$. We have

$$\begin{aligned}
J_h(w_h) &= \sum_{x \notin \Omega_{x_0}} \|D_h u_h(x)\|^2 + \|D_h w_h(x_0)\|^2 + \sum_{j=1}^d \|D_h w_h(x_0 + h_{x_0}^{r_j} r_j)\|^2 \\
&= \sum_{x \notin \Omega_{x_0}} \|D_h u_h(x)\|^2 + \sum_{i=1}^d \frac{1}{h_{x_0}^{-r_i}} (w_h(x_0) - w_h(x_0 - h_{x_0}^{-r_i} r_i))^2 \\
&\quad + \sum_{j=1}^d \sum_{i=1}^d \frac{1}{h^{-r_i}_{x_0 + h_{x_0}^{r_j}}} (w_h(x_0 + h_{x_0}^{r_j} r_j) - w_h(x_0 + h_{x_0}^{r_j} r_j - h_{x_0}^{-r_i} r_i))^2.
\end{aligned}$$

Next,

$$\begin{aligned}
&J_h(w_h) \\
&= \sum_{x \notin \Omega_{x_0}} \|D_h u_h(x)\|^2 + \sum_{j=1}^d \sum_{i=1\, i \neq j}^d \frac{1}{h^{-r_i}_{x_0 + h_{x_0}^{r_j}}} (w_h(x_0 + h_{x_0}^{r_j} r_j) \\
&\quad - w_h(x_0 + h_{x_0}^{r_j} r_j - h_{x_0}^{-r_i} r_i))^2 + \sum_{i=1}^d \frac{1}{h_{x_0}^{-r_i}} (w_h(x_0) - w_h(x_0 - h_{x_0}^{-r_i} r_i))^2 \\
&\quad + \frac{1}{h^{-r_i}_{x_0 + h_{x_0}^{r_i}}} (w_h(x_0 + h_{x_0}^{r_i} r_i) - w_h(x_0))^2.
\end{aligned}$$

We note that $h^{-r_i}_{x_0 + h_{x_0}^{r_i}} = h_{x_0}^{r_i}$ and we have

$$\begin{aligned}
&\frac{1}{h_{x_0}^{-r_i}} (w_h(x_0) - w_h(x_0 - h_{x_0}^{-r_i} r_i))^2 + \frac{1}{h_{x_0}^{r_i}} (w_h(x_0 + h_{x_0}^{r_i} r_i) - w_h(x_0))^2 \\
&= \frac{1}{h_{x_0}^{-r_i}} \big(u_h(x_0) - u_h(x_0 - h_{x_0}^{-r_i} r_i) + \frac{\epsilon}{4}\big)^2 + \frac{1}{h_{x_0}^{r_i}} \big(u_h(x_0 + h_{x_0}^{r_i} r_i) - u_h(x_0) - \frac{\epsilon}{4}\big)^2 \\
&= \frac{1}{h_{x_0}^{-r_i}} (u_h(x_0) - u_h(x_0 - h_{x_0}^{-r_i} r_i))^2 + \frac{1}{h_{x_0}^{r_i}} (u_h(x_0 + h_{x_0}^{r_i} r_i) - u_h(x_0))^2 \\
&\quad + \frac{\epsilon^2}{16} \Big(\frac{1}{h_{x_0}^{-r_i}} + \frac{1}{h_{x_0}^{r_i}} \Big) + \frac{\epsilon}{2 h_{x_0}^{-r_i}} (u_h(x_0) - u_h(x_0 - h_{x_0}^{-r_i} r_i)) \\
&\quad - \frac{\epsilon}{2 h_{x_0}^{r_i}} (u_h(x_0 + h_{x_0}^{r_i} r_i) - u_h(x_0)).
\end{aligned}$$

Thus, since for $i \neq j, w_h(x_0 + hr_j) - w_h(x_0 + hr_j - hr_i) = u_h(x_0 + hr_j) - u_h(x_0 + hr_j - hr_i)$, and by our choice of ϵ,

$$\begin{aligned}
J_h(w_h) &= J_h(u_h) + \frac{\epsilon^2}{16} \sum_{i=1}^d \Big(\frac{1}{h_{x_0}^{-r_i}} + \frac{1}{h_{x_0}^{r_i}} \Big) - \frac{\epsilon}{2} \sum_{i=1}^d \frac{h_{x_0}^{r_i} + h_{x_0}^{-r_i}}{2} \Delta_{r_i} u_h(x_0) \\
&\le J_h(u_h) + \frac{\epsilon \epsilon_0}{16} \sum_{i=1}^d h_{x_0}^{r_i} + h_{x_0}^{-r_i} - \frac{\epsilon}{4} \sum_{i=1}^d (h_{x_0}^{r_i} + h_{x_0}^{-r_i}) \Delta_{r_i} u_h(x_0) \\
&= J_h(u_h) + \frac{\epsilon}{4} \sum_{i=1}^d (h_{x_0}^{r_i} + h_{x_0}^{-r_i}) \big(\frac{\epsilon_0}{4} - \Delta_{r_i} u_h(x_0) \big) \\
&< J_h(u_h),
\end{aligned}$$

since $\Delta_e u_h(x_0) \geq \epsilon_0 > \epsilon_0/4$ for all $e \in V$. This contradicts the assumption that u_h is a minimizer and concludes the proof. $\square$

3. Convergence analysis

In this section, we first address the convergence of the solution u_h of (2.4) to the Aleksandrov solution u of (1.1). We require $V \subset \mathbb{Z}^d \backslash \{0\}$ to converge to $\mathbb{Z}^d \backslash \{0\}$. We thus simply assume that $V = \mathbb{Z}^d \setminus \{0\}$. We then give a direct proof of convergence to the viscosity solution of (1.1). Finally we make remarks about the case the right hand side of (2.4) approximates a sum of Dirac masses.

Definition 3.1. Let $u_h \in \mathcal{U}_h$ for each $h > 0$. We say that u_h converges to a function u uniformly on a compact set $K \subset \Omega$ if and only if for each sequence $h_k \to 0$ and for all $\epsilon > 0$, there exists $h_{-1} > 0$ such that for all h_k, $0 < h_k < h_{-1}$, we have

$$\max_{x \in \mathcal{N}_{h_k} \cap K} |u_{h_k}(x) - u(x)| < \epsilon.$$

3.1. Convergence to the Aleksandrov solution.

Theorem 3.2. *Let u_h solve (2.4). There is a subsequence u_{h_k} which converges uniformly on compact subsets to a convex function $v \in C(\overline{\Omega})$ such that*

$$\det D^2 v \leq f(x) \quad \text{in } \Omega$$
$$v = g \quad \text{on } \partial\Omega,$$

Proof. The family u_h is a uniformly bounded family of discrete convex functions by Theorem 2.13. Moreover $u_h = g$ on $\partial\Omega$ and $g \in C(\partial\Omega)$ can be extended to a convex function $\tilde{g} \in C(\overline{\Omega})$. In addition, by (2.3) and (2.7)

$$M_h[u_h](\Omega_h) \leq \sum_{x \in \Omega_h} M_V[u_h](x) = \sum_{x \in \Omega_h} h^d f(x) \leq A.$$

It is proven in [2, Theorem 14] that there is a subsequence u_{h_k} which converges uniformly on compact subsets to a convex function $v \in C(\overline{\Omega})$ such that $v = g$ on $\partial\Omega$. It is also proven in [2, Theorem 8] that $M_h[u_h]$ defines a Borel measure (as the Monge-Ampère measure of the convex envelope of u_h), which converges weakly to $M[v]$. Since by (2.3), we have

$$M_h[u_h](x) \leq h^d f(x), \quad x \in \Omega_h,$$

as an inequality in measures, we obtain $\det D^2 v \leq f(x)$. $\square$

To complete the proof we need additional notions. Given $u : \Omega \to \mathbb{R}$, the local subdifferential of u is given by

$$\partial_l u(x_0) = \big\{ p \in \mathbb{R}^d : \exists \text{ a neighborhood } U_{x_0} \text{ of } x_0 \text{ such that}$$
$$u(x) \geq u(x_0) + p \cdot (x - x_0), \text{ for all } x \in U_{x_0} \big\}.$$

Clearly for all $x_0 \in \Omega$ we have $\partial u(x_0) \subset \partial_l u(x_0)$. Moreover we the following result.

Lemma 3.3 ([14, Exercise 1])**.** *If Ω is convex and u is convex on Ω, then $\partial u(x) = \partial_l u(x)$ for all $x \in \Omega$.*

We recall that for a family of sets A_k

$$\limsup_k A_k = \cap_n \cup_{k \geq n} A_k.$$

Lemma 3.4. *Assume that $u_h \to v$ uniformly on compact subsets of Ω, with v convex and continuous. Then for $K \subset \Omega$ compact and any sequence $h_k \to 0$*

$$\limsup_{h_k \to 0} \partial_V u_{h_k}(K) \subset \partial v(K).$$

Proof. Let

$$p \in \limsup_{h_k \to 0} \partial_V u_{h_k}(K) = \cap_n \cup_{k \geq n} \partial_V u_{h_k}(K).$$

Thus for each n, there exists k_n and $x_{k_n} \in K \cap \mathbb{Z}^d_{h_{k_n}}$ such that $p \in \partial_V u_{h_{k_n}}(x_{k_n})$. By an abuse notation, let x_j denote a subsequence $x_{k_{n_j}}$ of x_{k_n} converging to $x_0 \in K$. We have $x_{k_{n_j}} \in K \cap \mathbb{Z}^d_{h_{k_{n_j}}}$ and so with our abuse of notation $x_j \in K \cap \mathbb{Z}^d_{h_j}$.

Let $B_\epsilon(x_0)$ denote the ball of center x_0 and radius ϵ in the maximum norm. We choose $\epsilon > 0$ such that $B_\epsilon(x_0) \subset \Omega$. Let $z \in B_{\epsilon/4}(x_0)$ and $z_{h_j} \in B_{\epsilon/4}(x_0) \cap \mathbb{Z}^d_{h_j}$ such that $z_{h_j} \to z$.

We have for j sufficiently large $\|x_j - x_0\| \leq \epsilon/8$. With $e = z_{h_j} - x_j, x_j + e = z_{h_j}$ while $x_j - e = 2x_j - z_{h_j} \in B_{\epsilon/4}(x_0)$ as $\|2x_j - z_{h_j} - x_0\| = \|2(x_j - x_0) + (x_0 - z_{h_j})\| \leq \epsilon/2$. That is $x_j \pm e \in \Omega \cap \mathbb{Z}^d_{h_j}$.

Since $p \in \partial_V u_{h_j}(x_j)$ for all j,

$$u_{h_j}(z_{h_j}) \geq u_{h_j}(x_j) + p \cdot (z_{h_j} - x_j). \tag{3.1}$$

Next, note that

$$|u_{h_j}(x_j) - v(x_0)| \leq |u_{h_j}(x_j) - v(x_j)| + |v(x_j) - v(x_0)|.$$

By the convergence of x_j to x_0 and the uniform convergence of u_h to v on K, we obtain $u_{h_j}(x_j) \to v(x_0)$ as $h_j \to 0$. Similarly $u_{h_j}(z) \to v(z)$ as $h_j \to 0$, using $u_{h_j}(z_{h_j}) - v(z) = (u_{h_j}(z_{h_j}) - v(z_{h_j})) + (v(z_{h_j}) - v(z))$. Taking pointwise limits in (3.1), we obtain

$$v(z) \geq v(x_0) + p \cdot (z - x_0) \ \forall z \in B_{\frac{\epsilon}{4}}(x_0).$$

We conclude that $p \in \partial_l v(K)$, the image of K by the local subdifferential of v, and thus $p \in \partial v(K)$ by Lemma 3.3, since v is convex and Ω convex. □

Theorem 3.5. *The limit convex function v given by Theorem 3.2 satisfies for $K \subset \Omega$ compact*

$$M[v](K) = \int_K f(x) dx.$$

Thus for any Borel set B, $M[v](B) = \int_B f(x) dx$.

Proof. It follows from Lemma 3.4

$$\limsup_{h_k \to 0} M_V[u_{h_k}](K) \leq M[v](K). \tag{3.2}$$

By Theorem 3.2 $M[v](K) \leq \int_K f(x) dx$. But $M_V[u_h](K) = \sum_{x \in K} h^d f(x)$. We conclude from (3.2) that

$$\limsup_{h_k \to 0} M_V[u_{h_k}](K) = \int_K f(x) dx \leq M[v](K) \leq \int_K f(x) dx,$$

from which the result follows. □

Theorem 3.6. *The solution u_h of* (2.4) *converges uniformly on compact subsets to the Aleksandrov solution u of* (1.1).

Proof. It follows from Theorems 3.2 and 3.5, that there is a subsequence which converges uniformly on compact subsets to a convex function $v \in C(\overline{\Omega})$ which solves (1.1). By unicity of the solution of the latter, the whole family must converge to u. □

3.2. **Convergence to the viscosity solution.** Again, we assume that $V = \mathbb{Z}^d \setminus \{0\}$. For a direct proof of convergence to the viscosity solution of (1.1), we recall the notion of monotonicity and consistency. We note that consistency was not used for the proof of convergence to the Aleksandrov solution.

Since we have proven convergence to the Aleksandrov solution, and Aleksandrov solutions are equivalent to viscosity solutions with our assumptions, we obtain convergence to the viscosity solution as well. The purpose of this section is to indicate how one may exploit equicontinuity to give a different proof of convergence to the viscosity solution.

The scheme (2.4) is said to be monotone if for z_h and w_h in $\mathcal{U}_h$, $z_h(y) \geq w_h(y), y \neq x$ with $z_h(x) = w_h(x)$, we have $M_h[z_h](x) \geq M_h[w_h](x)$.

We say that the scheme (2.4) is consistent if for all C^2 convex functions ϕ, a sequence $x_h \to x \in \Omega$

$$\lim_{h \to 0} \frac{1}{h^d} M_h[\phi](x_h) = \det D^2\phi(x).$$

We could only give a proof which is simple of a weaker form of consistency for the scheme (2.4). Let $B(x,r)$ denote the ball of center x and radius r in the maximum norm and let x_h' denote the unique mesh point in $B(x,h/2)$. We say that the scheme is weakly consistent if for all strictly convex quadratic polynomials ϕ, we have

$$\lim_{h \to 0} \frac{1}{h^d} M_h[\phi](x_h') = \det D^2\phi(x).$$

Lemma 3.7. *The scheme* (2.4) *is monotone.*

Proof. For z_h and w_h in $\mathcal{U}_h$ such that $z_h(y) \geq w_h(y), y \neq x$ with $z_h(x) = w_h(x)$, we have from the definition of discrete normal mapping $\partial_h w_h(x) \subset \partial_h z_h(x)$. Thus $M_h[z_h](x) \geq M_h[w_h](x)$, i.e. the scheme (2.4) is monotone. □

Lemma 3.8. *The scheme* (2.4) *is weakly consistent.*

Proof. Let $x \in \Omega$ and ϕ a strictly convex quadratic polynomial. Let μ have density $\det D^2\phi$, i.e. for each Borel set B, $\mu(B) = \int_B \det D^2\phi(x)dx$. Given a sequence of Borel measures μ_n which converges weakly to μ, we are interested in the uniform convergence of $\mu_n(B)$ to $\mu(B)$ for B in a subset $\mathcal{B}$ of Borel sets.

Let $\mathcal{B}$ consist of balls $B(x_0, r)$ of center x_0 and radius $r \leq r_0$ for $r_0 > 0$ fixed. Thus $\mathcal{B}$ consists of convex sets in a bounded set. By the Blaschke selection theorem, any sequence in $\mathcal{B}$ has a convergent subsequence, hence $\mathcal{B}$ is sequentially compact in the Hausdorff metric. By the same argument $\mathcal{B} \cap B$ is compact for any closed ball B. Since $\det D^2\phi > 0$, $\mu(\partial B(x_0,r)) = 0$, i.e. $\mathcal{B}$ is a μ-continuity class using the terminology of [8]. By [8, Theorem 6 and example 6], $\mathcal{B}$ is a μ-uniformity class, i.e. if μ_n weakly converges to μ, then $\mu_n(B(x_0,r)) \to \mu(B(x_0,r))$ uniformly in r.

We recall a form of the Moore-Osgood theorem on exchanging limits. Consider the double sequence $a_{n,k}$ with $a_{n,k} \to B_k$ uniformly in k as $n \to \infty$ and for each n, $\lim_{k\to\infty} a_{n,k} = A_n$. Then the double limits exist with

$$\lim_{n\to\infty} \lim_{k\to\infty} a_{n,k} = \lim_{k\to\infty} \lim_{n\to\infty} a_{n,k}.$$

We also recall that every continuity point of $\det D^2\phi$ is a Lebesgue point of $\det D^2\phi$. That is, as $\det D^2\phi \in C(\Omega)$ we have for $x_0 \in \Omega$

$$\lim_{r\to 0} \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} \det D^2\phi(x)dx = \det D^2\phi(x_0).$$

Let us denote as well by ϕ the restriction of ϕ to $\mathcal{N}_h$. By the weak convergence of $M_{h_k}[\phi]$ to μ, we have for $x\in\Omega$ and uniformly in r

$$\lim_{k\to\infty} M_{h_k}[\phi](B(x,r)) = \mu(B(x,r)).$$

Therefore,

$$\lim_{r\to 0}\frac{1}{|B(x,r)|}\lim_{k\to\infty} M_{h_k}[\phi](B(x,r)) = \lim_{r\to 0}\frac{1}{|B(x,r)|}\mu(B(x,r)) = \det D^2\phi(x).$$

By the Moore-Osgood theorem, we obtain

$$\lim_{k\to\infty}\lim_{r\to 0}\frac{1}{|B(x,r)|}M_{h_k}[\phi](B(x,r)) = \det D^2\phi(x).$$

We now take $r = h_k/2$ so that $|B(x,r)| = h_k^d$.

Now, there is a unique mesh point x'_{h_k} in $B(x,h_k)$ and $x'_{h_k}\to x$ as $k\to\infty$. Therefore $M_{h_k}[\phi](B(x,h_k/2)) = M_{h_k}[\phi](x'_{h_k})$ and we obtain the (weak) consistency of the scheme,

$$\lim_{k\to\infty}\frac{1}{h_k^d}M_{h_k}[\phi](x'_{h_k}) = \det D^2\phi(x). \qquad \square$$

By the stability of the scheme (2.4), c.f. Theorem 2.13, the half-relaxed limits

$$\begin{aligned} u^*(x) &= \limsup_{y\to x, h\to 0} u_h(y) = \lim_{\delta\to 0}\sup\{u_h(y) : y\in\Omega_h, |y-x|\le\delta, 0<h\le\delta\}\\ u_*(x) &= \liminf_{y\to x, h\to 0} u_h(y) = \lim_{\delta\to 0}\inf\{u_h(y) : y\in\Omega_h, |y-x|\le\delta, 0<h\le\delta\}, \end{aligned}$$

are well defined.

Our numerical experiments indicate that (2.4) is consistent. This is taken as an assumption in the following theorem. Rates of convergence of the Oliker-Prussner method, hence interior consistency, were given in [27]. Similar arguments could be followed for the discretization (2.4). But this would make this contribution too long.

Theorem 3.9. *Let $f > 0$ and $f \in C(\overline{\Omega})$. Assume that g can be extended to a convex function $\tilde{g}\in C(\overline{\Omega})$. Under the assumption that (2.4) is consistent, the upper half-relaxed limit u^* is a viscosity sub solution of $\det D^2u(x) = f(x)$ and the lower half-relaxed limit u_* is a viscosity super solution of $\det D^2u(x) = f(x)$ at every point x of Ω. Moreover, solutions u_h of (2.4) converge uniformly on compact subsets to the unique viscosity solution of (1.1).*

Proof. The result follows from the results of [5] and the stability, consistency and monotonicity of the scheme, combined with equicontinuity of the approximations. The part of the proof below which uses the consistency and monotonicity of the scheme follows [9].

The family u_h is a family of discrete convex functions which is uniformly bounded and by (2.7) have Monge-Ampère masses uniformly bounded (using the terminology of [2]). Moreover $u_h = g$ on $\partial\Omega_h$ for a convex function $g\in C(\overline{\Omega})$. By [2, Theorem 14], there is a subsequence u_{h_k} which converges uniformly on compact subsets to a convex function $v \in C(\overline{\Omega})$ which solves $v = g$ on $\partial\Omega$. It follows from the definitions that $v = u^* = u_*$ on Ω. At this point, it is not known yet that the limit convex function v is a viscosity solution of (1.1).

We show that $v = u_*$ is a viscosity super solution of $\det D^2 u(x) = f(x)$ at every point x of Ω. Recall that $v \in C(\overline{\Omega})$. Let $x_0 \in \Omega$ and ϕ be a strictly convex quadratic polynomial such that $u_* - \phi$ has a local minimum at x_0 with $(u_* - \phi)(x_0) = 0$. Without loss of generality, we may assume that x_0 is a strict local minimum.

Let B_0 denote a closed ball contained in Ω and containing x_0 in its interior. We let x_{h_l} be a subsequence in B_0 such that $x_{h_l} \to x_0$ with $u_{h_l}(x_l) \to u_*(x_0)$. Let x_l' be defined by

$$c_l := (u_{h_l} - \phi)(x_l') = \min_{B_0} u_{h_l} - \phi.$$

Since the sequence x_l' is bounded, it converges to some x_1 after possibly passing to a subsequence. Since $(u_{h_l} - \phi)(x_l') \le (u_{h_l} - \phi)(x_{h_l})$ we have

$$(u_* - \phi)(x_0) = \lim_{l\to\infty}(u_{h_l} - \phi)(x_{h_l}) \ge \liminf_{l\to\infty}(u_{h_l} - \phi)(x_l') \ge (u_* - \phi)(x_1).$$

Since x_0 is a strict minimizer of the difference $u_* - \phi$, we conclude that $x_0 = x_1$ and $c_l \to 0$ as $l \to \infty$.

By definition

$$u_{h_l}(x) \ge \phi(x) + c_l, \quad \forall x \in B_0 \cap \Omega_{h_l},$$

with equality at $x = x_l'$, and thus, by the monotonicity of the scheme

$$0 = \frac{1}{h_l^d} M_{h_l}[u_{h_l}](x_l') - f(x_l') \ge \frac{1}{h_l^d} M_{h_l}[\phi + c_l](x_l') - f(x_l') = \frac{1}{h_l^d} M_{h_l}[\phi](x_l') - f(x_l'),$$

which gives by the consistency of the scheme $\det D^2\phi(x_0) - f(x_0) \le 0$.

Similarly one shows that if ϕ is a strictly convex quadratic polynomial such that $u^* - \phi$ has a local maximum at x_0 with $(u^* - \phi)(x_0) = 0$, we have $\det D^2\phi(x_0) - f(x_0) \ge 0$.

It follows that $v = u^* = u_*$ on Ω is a viscosity solution of $\det D^2 u = f$. By the comparison principle Theorem 2.7, v is equal to the unique viscosity solution of (1.1). Thus all subsequences u_{h_k} converge uniformly on compact subsets to the same limit. This concludes the proof. □

Several discrete Monge-Ampère equations, e.g. [6, 22], can be written as

$$\mathcal{M}_h[u_h](x) = h^d f(x), x \in \Omega_h,$$

for some operator $\mathcal{M}_h$ which satisfies

$$M_h[u_h] \le C\mathcal{M}_h[u_h].$$

For $f \in C(\overline{\Omega}), \sum_{x\in\Omega_h} h^d f(x) \to \int_\Omega f(x)dx$ and thus

$$\sum_{x\in\Omega_h} M_h[u_h](x) \le C \sum_{x\in\Omega_h} \mathcal{M}_h[u_h](x) = C \sum_{x\in\Omega_h} h^d f(x) \le A,$$

for a constant A independent of h. Thus (2.7) holds for schemes such as the ones in [6, 22] and convergence to the viscosity solution on convex domains not necessarily strictly convex can be proven as for Theorem 3.9.

3.3. **Remarks on the case of a sum of Dirac masses for the right-hand side.** Consider the problem

$$\begin{aligned} \det D^2 u &= \sum_{l=1}^{N} c_l \delta_{d_l} \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{3.3}$$

where d_l is a mesh point, c_l a real number, δ_{d_l} the Dirac mass at d_l and N is the number of Dirac masses. Here, we assume that the parameter h is chosen so that d_l is a mesh point. For example, we may restrict h to take the form $1/2^j$ for a positive integer j when Ω is a cube. The corresponding discrete problem is

$$\begin{aligned} M_V[u_h](x) &= f_h(x), x \in \Omega_h \\ u_h(x) &= g(x), x \in \partial\Omega_h, \end{aligned} \tag{3.4}$$

where f_h is a mesh function which equals 0 at all mesh points, except d_l where it takes values c_l for $l = 1, \dots, N$. Discrete solutions have Monge-Ampère masses uniformly bounded as $\sum_{x\in\Omega_h} f_h(x)$ is uniformly bounded. Our convergence analysis to the Aleksandrov solution holds in this case as well. This does not appear to be an effective method in the case of Dirac masses.

The method we discussed is a variant of the Oliker-Prussner method [28]. For the latter (3.3) requires the masses c_l to be non zero. But with our approximations f_h, $f_h(x) = 0$ in parts of the computational domain Ω_h. A method which is solely based on the interpretation of $f(x)$ as a continuous function should handle better the case where $f_h(x) = 0$ in parts of the computational domain. The numerical method discussed in [7] can be interpreted as a variant of (3.4) with numerical integration for the computation of the discrete subdifferential at points d_l and the use of a different scheme elsewhere.

4. Numerical experiments

The computational domain is the unit square $[0,1]^2$. The initial guess for the iterations was taken as a shifted quadratic $x^2 + y^2 - 2$. The discrete nonlinear system was solved with a damped Newton's method [22]. Let $\delta, \rho \in (0,1)$. Given an initial guess u_h^0 to the nonlinear equations $G(u_h) = 0$, set $k = 0$. If $G(u_h^k) = 0$, stop. Put $p^k(\tau) = u_h^k - \tau G'(u_h^k)^{-1} G(u_h^k)$ and let i_k be the smallest non-negative integer i such that

$$\|G(p^k(\rho^i))\| \le (1 - \delta\rho^i)\|G(v_h^k)\|.$$

We set $u_h^{k+1} = p^k(\rho^{i_k})$. In the experiments, we take $\delta = \rho = 1$. Errors are given in the maximal norm. We used a 1.4 GHz Quad-Core Intel Core i5 MacBook Pro and the implementation was in Matlab.

Table 1. Smooth solution $u(x,y) = e^{(x^2+y^2)/2}$ with $g(x,y) = e^{(x^2+y^2)/2}$ and $f(x,y) = (1 + x^2 + y^2)e^{x^2+y^2}$.

h	$1/2^4$	$1/2^5$	$1/2^6$	$1/2^7$	$1/2^8$	$1/2^9$
Error	$3.51\ 10^{-4}$	$8.81\ 10^{-5}$	$2.20\ 10^{-5}$	$5.51\ 10^{-6}$	$1.38\ 10^{-6}$	$3.45\ 10^{-7}$
Rate		1.99	1.99	1.99	1.99	2
Time	$7.97\ 10^{-2}$	$2.01\ 10^{-1}$	$1.01\ 10^{-0}$	$4.27\ 10^{-0}$	$2.02\ 10^{+1}$	$9.75\ 10^{+1}$

Table 2. Non smooth solution $u(x, y) = -\sqrt{2 - x^2 - y^2}$ with $g(x, y) = -\sqrt{2 - x^2 - y^2}$ and $f(x, y) = 2/(2 - x^2 - y^2)^2$.

h	$1/2^4$	$1/2^5$	$1/2^6$	$1/2^7$	$1/2^8$	$1/2^9$
Error	$6.89\ 10^{-4}$	$2.36\ 10^{-4}$	$8.21\ 10^{-5}$	$2.88\ 10^{-5}$	$1.01\ 10^{-5}$	$3.58\ 10^{-6}$
Rate		1.55	1.52	1.51	1.50	1.50
Time	$7.07\ 10^{-2}$	$1.94\ 10^{-1}$	$2.17\ 10^{-0}$	$1.94\ 10^{+1}$	$1.84\ 10^{+1}$	$1.91\ 10^{+3}$

For the smooth solution, the number of Newton iterations was about 40, while for the non smooth solution it took hundreds of iterations for fine resolutions.

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