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BASISNESS OF FUČÍK EIGENFUNCTIONS FOR THE DIRICHLET LAPLACIAN

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Abstract. We provide improved sufficient assumptions on sequences of Fučík eigenvalues of the one-dimensional Dirichlet Laplacian which guarantee that the corresponding Fučík eigenfunctions form a Riesz basis in $L^2(0,\pi)$. For that purpose, we introduce a criterion for a sequence in a Hilbert space to be a Riesz basis.

1. Introduction

We study basis properties of sequences of eigenfunctions of the *Fučík eigenvalue problem* for the one-dimensional Dirichlet Laplacian

$$
\begin{gathered}
-u''(x) = \alpha u^+(x) - \beta u^-(x), \quad x \in (0,\pi), \\
u(0) = u(\pi) = 0,
\end{gathered} \tag{1.1}
$$

where $u^+ = \max(u,0)$ and $u^- = \max(-u,0)$. The Fučík spectrum is the set $\Sigma(0,\pi)$ of pairs $(\alpha,\beta) \in \mathbb{R}^2$ for which (1.1) possesses a nontrivial classical solution. Any $(\alpha,\beta) \in \Sigma(0,\pi)$ is called *Fučík eigenvalue* and any corresponding nontrivial classical solution of (1.1) is called *Fučík eigenfunction.* The Fučík eigenvalue problem (1.1) was introduced in [4] and [6] to study elliptic equations with "jumping" nonlinearities, and it has since been widely investigated in various aspects and for different operators, see, e.g., the surveys [3], [8, Chapter 9.4], and references therein. To the best of our knowledge, basisness of sequences of Fučík eigenfunctions was considered for the first time in [2]. In that article, we provided several sufficient assumptions on sequences of Fučík eigenvalues to obtain Riesz bases of $L^2(0,\pi)$ consisting of Fučík eigenfunctions. Let us recall that a sequence is a Riesz basis in a Hilbert space if it is the image of an orthonormal basis of that space under a linear homeomorphism, see, e.g., [9]. The aim of the present note is to use more general techniques to significantly improve the results of [2].

Let us describe the structure of the Fučík spectrum $\Sigma(0,\pi)$. It is not hard to see that the lines $\{1\} \times \mathbb{R}$ and $\mathbb{R} \times \{1\}$ are subsets of $\Sigma(0,\pi)$, since they correspond to sign-constant solutions of (1.1) which are constant multiples of $\sin x$, the first eigenfunction of the Dirichlet Laplacian in $(0,\pi)$. The remaining part of $\Sigma(0,\pi)$ is

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exhausted by the hyperbola-type curves

$$\Gamma_n = \Big\{(\alpha,\beta) \in \mathbb{R}^2 : \frac{n}{2}\frac{\pi}{\sqrt{\alpha}} + \frac{n}{2}\frac{\pi}{\sqrt{\beta}} = \pi\Big\}$$

for even $n \in \mathbb{N}$, and

$$\Gamma_n = \Big\{(\alpha,\beta) \in \mathbb{R}^2 : \frac{n+1}{2}\frac{\pi}{\sqrt{\alpha}} + \frac{n-1}{2}\frac{\pi}{\sqrt{\beta}} = \pi\Big\},$$
$$\widetilde{\Gamma}_n = \Big\{(\alpha,\beta) \in \mathbb{R}^2 : \frac{n-1}{2}\frac{\pi}{\sqrt{\alpha}} + \frac{n+1}{2}\frac{\pi}{\sqrt{\beta}} = \pi\Big\}$$

for odd $n \geq 3$, see, e.g., [6, Lemma 2.8]. Evidently, $(\alpha,\beta) \in \Gamma_n$ for odd $n \geq 3$ implies $(\beta,\alpha) \in \widetilde{\Gamma}_n$. If u is a Fučík eigenfunction for some (α,β), then so is tu for any $t > 0$, while $-tu$ is a Fučík eigenfunction for (β,α). Hence, we neglect the curve $\widetilde{\Gamma}_n$ from our investigation of the basis properties of Fučík eigenfunctions. Each sign-changing Fučík eigenfunction consists of alternating positive and negative bumps, where positive bumps are described by $C_1 \sin(\sqrt{\alpha}(x - x_1))$, while negative bumps are described by $C_2 \sin(\sqrt{\beta}(x - x_2))$, for proper constants $C_1, C_2, x_1, x_2 \in \mathbb{R}$.

We want to uniquely specify a Fučík eigenfunction for each point of $\Sigma(0,\pi)$. In slight contrast to [2], we normalize Fučík eigenfunctions in such a way that they are "close" to the functions

$$\varphi_k(x) = \sqrt{\frac{2}{\pi}}\,\sin(kx), \quad k \in \mathbb{N},$$

which form a complete *orthonormal* system in $L^2(0,\pi)$. This choice will be helpful in the proof of our main result, Theorem 1.3, below.

Definition 1.1. Let $n \geq 2$ and $(\alpha,\beta) \in \Gamma_n$. The *normalized Fučík eigenfunction* $g^n_{\alpha,\beta}$ is the C^2-solution of the boundary value problem (1.1) with $(g^n_{\alpha,\beta})'(0) > 0$ and which is normalized by

$$\|g^n_{\alpha,\beta}\|_\infty = \sup_{x\in[0,\pi]} |g^n_{\alpha,\beta}(x)| = \sqrt{\frac{2}{\pi}}.$$

For $n = 1$, we set $g^1_{\alpha,\beta} = \varphi_1$ for every $(\alpha,\beta) \in (\{1\} \times \mathbb{R}) \cup (\mathbb{R} \times \{1\})$.

Piecewise definitions of the Fučík eigenfunctions $f^n_{\alpha,\beta} = \sqrt{\pi/2}\, g^n_{\alpha,\beta}$ can be found in the equations (1.2) and (1.3) in [2]. In accordance to [2], we study the basisness of sequences of Fučík eigenfunctions described by the following definition.

Definition 1.2. We define the *Fučík system* $G_{\alpha,\beta} = \{g^n_{\alpha(n),\beta(n)}\}$ as a sequence of normalized Fučík eigenfunctions with mappings $\alpha,\beta\colon \mathbb{N} \to \mathbb{R}$ satisfying $\alpha(1) = \beta(1) = 1$ and $(\alpha(n),\beta(n)) \in \Gamma_n$ for every $n \geq 2$.

We can now formulate our main result on the basisness of Fučík systems which presents a non-trivial generalization of [2, Theorems 1.4 and 1.9].

Theorem 1.3. *Let $G_{\alpha,\beta}$ be a Fučík system. Let N be a subset of the even natural numbers and $N_* = \mathbb{N} \setminus N$. Assume that*

$$\sum_{n\in N_*} \Big[1 - \frac{\langle g^n_{\alpha,\beta}, \varphi_n\rangle^2}{\|g^n_{\alpha,\beta}\|^2}\Big] + E^2\Big(\sup_{n\in N}\Big\{\frac{4\max(\alpha(n),\beta(n))}{n^2}\Big\}\Big) < 1, \tag{1.2}$$

with $\sup_{n\in N}\left\{4\max(\alpha(n),\beta(n))/n^2\right\}\in[4,9)$. *Here,* $E:[4,9)\to\mathbb{R}$ *is a strictly increasing function defined as*

$$\begin{aligned} E(\gamma) &= \frac{2\sqrt{2}}{\pi}\frac{\gamma^2}{\sqrt{\gamma}-1}\frac{(\sqrt{\gamma}-2)\sin\left(\frac{\pi}{\sqrt{\gamma}}\right)}{(\gamma-1)(2\sqrt{\gamma}-1)} \\ &\quad+\frac{((3+\pi^2)\gamma+(9-2\pi^2)\sqrt{\gamma}-6)(\sqrt{\gamma}-2)}{3(\sqrt{\gamma}-1)(\sqrt{\gamma}+2)(3\sqrt{\gamma}-2)} \\ &\quad+\frac{4}{\sqrt{3}\pi}\frac{\gamma^2}{\sqrt{\gamma}-1}\frac{(\sqrt{\gamma}-2)\sin\left(-\frac{3\pi}{\sqrt{\gamma}}\right)}{(9-\gamma)(2\sqrt{\gamma}-3)(4\sqrt{\gamma}-3)} \\ &\quad+\frac{2}{\pi}\frac{\gamma^2}{\sqrt{\gamma}-1}\frac{(\sqrt{\gamma}-2)}{(16-\gamma)(3\sqrt{\gamma}-4)(5\sqrt{\gamma}-4)} \\ &\quad+\sqrt{\frac{6}{5}}\frac{2}{\pi}\frac{\gamma^2(\sqrt{\gamma}-2)}{\sqrt{\gamma}-1}\sum_{k=5}^{\infty}\frac{1}{(k^2-\gamma)((k-1)\sqrt{\gamma}-k)((k+1)\sqrt{\gamma}-k)}. \end{aligned} \tag{1.3}$$

Then $G_{\alpha,\beta}$ *is a Riesz basis in* $L^2(0,\pi)$.

The proof of this theorem is given in Section 3 and is based on a general basisness criterion provided in Section 2. We visualize special cases of domains on the (α,β)-plane described in Theorem 1.3 in Figures 1 and 2 below.

Notice that, thanks to the orthonormality of $\{\varphi_n\}$, the terms in the first sum in (1.2) satisfy

$$0\le 1-\frac{\langle g^n_{\alpha,\beta},\varphi_n\rangle^2}{\|g^n_{\alpha,\beta}\|^2}=\|g^n_{\alpha,\beta}-\varphi_n\|^2-\frac{(\|g^n_{\alpha,\beta}\|^2-\langle g^n_{\alpha,\beta},\varphi_n\rangle)^2}{\|g^n_{\alpha,\beta}\|^2}\le\|g^n_{\alpha,\beta}-\varphi_n\|^2, \tag{1.4}$$

and we have the explicit bounds

$$\|g^n_{\alpha,\beta}-\varphi_n\|^2\le\begin{cases}\frac{8(3+\pi^2)}{9}\frac{(\max(\sqrt{\alpha},\sqrt{\beta})-n)^2}{n^2} & \text{for even } n,\\ \frac{8n^2(n^2+1)}{(n-1)^4}\frac{(\sqrt{\alpha}-n)^2}{n^2} & \text{for odd } n\ge3 \text{ with } \alpha\ge n^2,\\ \frac{10n^2(n^2+1)}{(n+1)^4}\frac{(\sqrt{\beta}-n)^2}{n^2} & \text{for odd } n\ge3 \text{ with } \beta>n^2,\end{cases} \tag{1.5}$$

see the estimates (3.2), (3.4), (3.5), (3.6) in [2, Section 3]. In view of (1.4), if we choose $N=\emptyset$, then Theorem 1.3 is an improvement of [2, Theorem 1.4].

Let us summarize a few properties of the function E defined in Theorem 1.3, see the end of Section 3 for a discussion.

Lemma 1.4. *The function* E *has the following properties:*

(i) E *is continuous in* $[4,9)$.
(ii) *Each summand in the definition* (1.3) *of* E *is strictly increasing in* $[4,9)$.
(iii) *We have* $E(4)=0$ *and* $E(6.49278\dots)=1$.
(iv) *The infinite sum in the definition* (1.3) *of* E *in* $(4,9)$ *can be expressed as follows:*

$$\begin{aligned}&\sqrt{\frac{6}{5}}\frac{2}{\pi}\frac{\gamma^2(\sqrt{\gamma}-2)}{\sqrt{\gamma}-1}\sum_{k=5}^{\infty}\frac{1}{(k^2-\gamma)((k-1)\sqrt{\gamma}-k)((k+1)\sqrt{\gamma}-k)}\\ &=\sqrt{\frac{6}{5}}\frac{2}{\pi}\frac{\sqrt{\gamma}}{\sqrt{\gamma}-1}\sum_{k=5}^{\infty}\Big(\frac{1}{k^2-\gamma}-\frac{1}{k^2-\frac{\gamma}{(\sqrt{\gamma}-1)^2}}\Big)\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{6}{5}} \frac{1}{\pi(\sqrt{\gamma}-1)} \Big(\pi(\sqrt{\gamma}-1)\cot\Big(\frac{\pi\sqrt{\gamma}}{\sqrt{\gamma}-1}\Big) - \pi\cot(\pi\sqrt{\gamma}) - (\sqrt{\gamma}-2)\Big) \\
&\quad - \sqrt{\frac{6}{5}} \frac{2}{\pi} \frac{\gamma^2(\sqrt{\gamma}-2)}{\sqrt{\gamma}-1} \sum_{k=1}^{4} \frac{1}{(k^2-\gamma)((k-1)\sqrt{\gamma}-k)((k+1)\sqrt{\gamma}-k)}.
\end{aligned}
$$

The interval $[4, 9)$ appears naturally in the proof of Theorem 1.3. In fact, Lemma 1.4 (iii) indicates that the highest possible value of $\sup_{n\in N} \{4\max(\alpha(n), \beta(n))/n^2\}$ to satisfy the assumption (1.2) is even smaller than 9.

We obtain the following practical corollary of Theorem 1.3 by applying the upper bounds (1.5) for the case that N is the set of all even natural numbers, see Figure 1.

Corollary 1.5. *Let $G_{\alpha,\beta}$ be a Fučík system, and $\varepsilon > 0$. Assume that*

$$
\sup_{n\in\mathbb{N}\ even} \Big\{\frac{4\max(\alpha(n), \beta(n))}{n^2}\Big\} < 6.49278\dots
$$

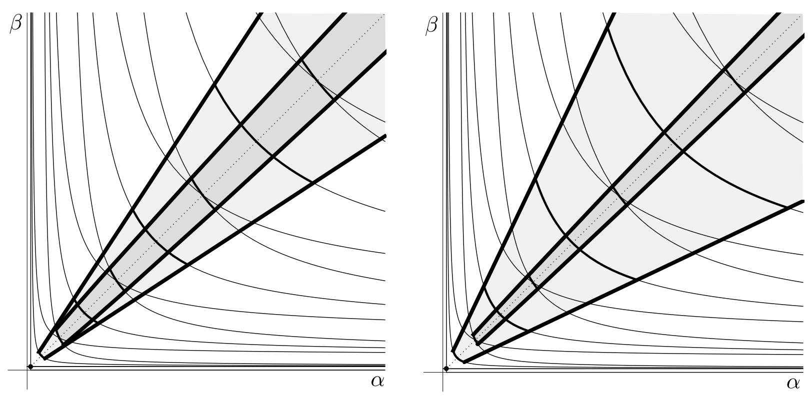
and

$$
\max(\alpha(n), \beta(n)) \le \Big(n + \sqrt{c_n}\, n^{(1-\varepsilon)/2}\Big)^2 \quad \textit{for all odd } n \ge 3,
$$

where

$$
0 \le c_n < \frac{1 - E^2\Big(\sup_{n\in\mathbb{N}\ even} \Big\{\frac{4\max(\alpha(n),\beta(n))}{n^2}\Big\}\Big)}{45\left(\left(1 - \frac{1}{2^{1+\varepsilon}}\right)\zeta(1+\varepsilon) - 1\right)}
$$

with the Riemann zeta function ζ. Then $G_{\alpha,\beta}$ is a Riesz basis in $L^2(0,\pi)$.



(A) (B)

Figure 1. The assumptions of Corollary 1.5 are satisfied for $(\alpha(n), \beta(n))$ belonging to bold parts of curves Γ_n inside the shaded regions. We have $\varepsilon = 0.5$ for both panels and $\sup_{n\in\mathbb{N}\text{ even}} \big\{\frac{4\max(\alpha(n),\beta(n))}{n^2}\big\} = 5,\ 6$ in panel (A), (B), respectively.

If we assume that the first sum of (1.2) in Theorem 1.3 is vanishing, which corresponds to $c_n = 0$ for all odd $n \ge 3$ in the previous corollary, we obtain the following result.

Corollary 1.6. *Let $G_{\alpha,\beta}$ be a Fučík system such that $g^n_{\alpha,\beta} = \varphi_n$ for any odd n. Assume that*

$$\sup_{n\in\mathbb{N}\ even} \Big\{\frac{4\max(\alpha(n),\beta(n))}{n^2}\Big\} < 6.49278\ldots \tag{1.6}$$

Then $G_{\alpha,\beta}$ is a Riesz basis in $L^2(0,\pi)$.

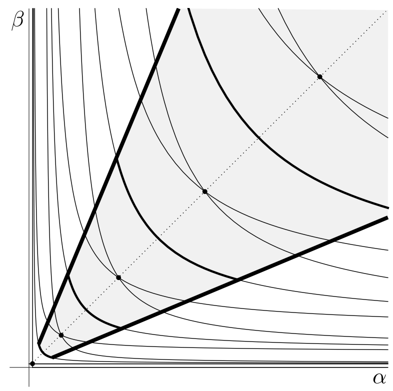


FIGURE 2. The assumption (1.6) is satisfied for $(\alpha(n),\beta(n))$ belonging to bold parts of curves Γ_n inside the shaded region.

We remark that Corollaries 1.5 and 1.6 are significant improvements of [2, Theorem 1.9] since each point $(\alpha(n),\beta(n)) \in \Gamma_n$ for even $n \geq 2$ is free to belong to the whole angular sector in between the line

$$\beta = \Big(\sqrt{\sup_{n\in\mathbb{N}\,\text{even}} \Big\{\frac{4\max(\alpha(n),\beta(n))}{n^2}\Big\}} - 1\Big)^{-2}\alpha$$

and its reflection with respect to the main diagonal $\alpha = \beta$, and the angle of that sector is allowed to be larger than the one provided by [2, Theorem 1.9]. We refer to Figure 2 for the domain on the (α,β)-plane given by Corollary 1.6. Moreover, Corollary 1.5 improves [2, Theorem 1.9] in the sense that $g^n_{\alpha,\beta}$ for odd $n \geq 3$ might differ from φ_n, see Figure 1.

2. Basisness criterion

In this section, we formulate a useful generalization of the separation of variables approach of [5] in a real Hilbert space X. The provided criterion will be applied to the space $L^2(0,\pi)$ to prove our main result, Theorem 1.3, in the subsequent section.

Theorem 2.1. *Let $M \in \mathbb{N}$. Let $N_*, N_m \subset \mathbb{N}$, $1 \leq m \leq M$, be pairwise disjoint sets which form a decomposition of the natural numbers, i.e.,*

$$N_* \cup \bigcup_{m=1}^{M} N_m = \mathbb{N}.$$

Let $\{\phi_n\}$ be a complete orthonormal sequence in X and $\{f_n\} \subset X$ be a sequence that can be represented as

$$f_n = \phi_n + \sum_{k=1}^{\infty} C^m_{n,k} T^m_k \phi_n \quad \text{for every } n \in N_m,\ 1 \leq m \leq M, \tag{2.1}$$

and satisfies

$$\Lambda_* := \Big(\sum_{n\in N_*} \Big[1 - \frac{\langle f_n, \phi_n\rangle^2}{\|f_n\|^2}\Big]\Big)^{1/2} < \infty.$$

In the representation formula (2.1), $\{T_k^m\}$ is a family of bounded linear mappings from X to itself with bounds $\|T_k^m\|_* \le t_k^m$ on the operator norm and $\{C_{n,k}^m\}$ is a family of constants with uniform bounds $|C_{n,k}^m| \le c_k^m$ that satisfy

$$\Lambda_m := \sum_{k=1}^{\infty} c_k^m t_k^m < \infty. \tag{2.2}$$

Then $\{f_n\}$ is a basis in X provided that

$$\Lambda_*^2 + \sum_{m=1}^{M} \Lambda_m^2 < 1. \tag{2.3}$$

If, in addition, the subsequence $\{f_n\}_{n\in N_*}$ is bounded, then $\{f_n\}$ is a Riesz basis in X.

Proof. Denote $\widetilde{f}_n = \rho_n f_n$, where $\rho_n = 1$ for $n \in \mathbb{N} \setminus N_*$, and the values of ρ_n for $n \in N_*$ will be specified later. Let $\{a_n\}_{n\in\widetilde{N}}$ be an arbitrary finite sequence of constants with a finite index set $\widetilde{N} \subset \mathbb{N}$. Setting $\widetilde{N}_* = N_* \cap \widetilde{N}$ and $\widetilde{N}_m = N_m \cap \tilde{N}$ for every $1 \le m \le M$, we obtain

$$\big\| \sum_{n\in\widetilde{N}} a_n(\widetilde{f}_n - \phi_n)\big\| \le \sum_{m=1}^{M} \big\| \sum_{n\in\widetilde{N}_m} a_n(f_n - \phi_n)\big\| + \big\| \sum_{n\in\widetilde{N}_*} a_n(\rho_n f_n - \phi_n)\big\|. \tag{2.4}$$

For the first sum on the right-hand side of (2.4), we apply the representation (2.1) and obtain

$$\begin{aligned}
&\sum_{m=1}^{M} \big\| \sum_{n\in\widetilde{N}_m} a_n(f_n - \phi_n)\big\| \\
&= \sum_{m=1}^{M} \big\| \sum_{n\in\widetilde{N}_m} a_n \sum_{k=1}^{\infty} C_{n,k}^m T_k^m \phi_n\big\| = \sum_{m=1}^{M} \big\| \sum_{k=1}^{\infty} T_k^m \sum_{n\in\widetilde{N}_m} C_{n,k}^m a_n \phi_n\big\| \\
&\le \sum_{m=1}^{M} \sum_{k=1}^{\infty} \big\|T_k^m \sum_{n\in\widetilde{N}_m} C_{n,k}^m a_n \phi_n\big\| \le \sum_{m=1}^{M} \sum_{k=1}^{\infty} t_k^m \big\| \sum_{n\in\widetilde{N}_m} C_{n,k}^m a_n \phi_n\big\| \\
&\le \sum_{m=1}^{M} \sum_{k=1}^{\infty} t_k^m c_k^m \big\| \sum_{n\in\widetilde{N}_m} a_n \phi_n\big\| = \sum_{m=1}^{M} \Lambda_m \big\| \sum_{n\in\widetilde{N}_m} a_n \phi_n\big\|,
\end{aligned}$$

while for the second sum we obtain

$$\big\| \sum_{n\in\widetilde{N}_*} a_n(\rho_n f_n - \phi_n)\big\| \le \Big(\sum_{n\in\widetilde{N}_*} \|\rho_n f_n - \phi_n\|^2\Big)^{1/2} \Big(\sum_{n\in\widetilde{N}_*} |a_n|^2\Big)^{1/2}.$$

Let us choose ρ_n to be a minimizer of the distance $\|\rho f_n - \phi_n\|^2$ with respect to ρ. Since

$$\|\rho f_n - \phi_n\|^2 = \rho^2 \|f_n\|^2 - 2\rho\langle f_n, \phi_n\rangle + 1,$$

we readily see that

$$\|\rho_n f_n - \phi_n\|^2 = \min_{\rho\in\mathbb{R}} \|\rho f_n - \phi_n\|^2 = 1 - \frac{\langle f_n, \phi_n\rangle^2}{\|f_n\|^2} = \|f_n - \phi_n\|^2 - \frac{(\|f_n\|^2 - \langle f_n, \phi_n\rangle)^2}{\|f_n\|^2}$$

with $\rho_n = \langle f_n, \phi_n\rangle / \|f_n\|^2$. Evidently, we have $|\rho_n| \le 1$. We remark that in case of $\rho_n = 0$, we get $\Lambda_* \ge 1$ which violates the assumption (2.3). Applying now the Cauchy inequality, we deduce from (2.4) that

$$\begin{aligned} \Big\| \sum_{n\in\widetilde{N}} a_n(\widetilde{f}_n - \phi_n)\Big\| &\le \sum_{m=1}^{M} \Lambda_m \Big\| \sum_{n\in\widetilde{N}_m} a_n\phi_n\Big\| + \Lambda_* \Big(\sum_{n\in\widetilde{N}_*} |a_n|^2\Big)^{1/2} \\ &\le \Big(\sum_{m=1}^{M} \Lambda_m^2 + \Lambda_*^2\Big)^{1/2} \Big\| \sum_{n\in\widetilde{N}} a_n\phi_n\Big\|. \end{aligned}$$

We conclude from the assumption (2.3) that the sequence $\{\widetilde{f}_n\}$ is Paley-Wiener near to the complete orthonormal sequence $\{\phi_n\}$ and, thus, it is a Riesz basis in X, see, e.g., [9, Chapter 1, Theorem 10]. Clearly, $\{f_n\} = \{\rho_n^{-1}\widetilde{f}_n\}$ is a basis in X. Assume that the subsequence $\{f_n\}_{n\in N_*}$ is bounded. Then there exists $0 < c < 1$ such that $|\rho_n| \ge c$ for all $n \in \widetilde{N}_*$. This is evident for finite N_* since $\rho_n \ne 0$. In the case of infinite N_*, if we suppose that ρ_n goes to zero up to a subsequence, then the sum

$$\Lambda_* = \Big(\sum_{n\in N_*} \Big[1 - \frac{\langle f_n, \phi_n\rangle^2}{\|f_n\|^2}\Big]\Big)^{1/2} = \Big(\sum_{n\in N_*} \Big[1 - \rho_n^2\|f_n\|^2\Big]\Big)^{1/2}$$

does not converge. Recalling $\rho_n = 1$ for every $n \in \mathbb{N}\setminus N_*$, we obtain $1 \le |\rho_n^{-1}| \le c^{-1}$ for all $n \in \mathbb{N}$ which implies that $\{f_n\}$ is a Riesz basis in X, see, e.g., [9, Chapter 1, Theorem 9]. □

In the case $N_1 = \mathbb{N}$, Theorem 2.1 simplifies to Theorem D from [5] and for $N_* = \mathbb{N}$ we get the result of Theorem V-2.21 and Corollary V-2.22 i) from [7] which were discussed in [2].

Remark 2.2. It can be seen from the proof of Theorem 2.1 that if we weaken the definition of Λ_* to

$$\widetilde{\Lambda}_* := \Big(\sum_{n\in N_*} \|f_n - \phi_n\|^2\Big)^{1/2} \le \Lambda_*,$$

then we can formulate the following result under the assumptions of Theorem 2.1: the sequence $\{f_n\}$ is a Riesz basis in X provided that

$$\widetilde{\Lambda}_*^2 + \sum_{m=1}^{M} \Lambda_m^2 < 1.$$

The boundedness of the subsequence $\{f_n\}_{n\in N_*}$ is not required under this modified assumption.

3. Proof of Theorem 1.3

We prove Theorem 1.3 by applying the general basisness criterion introduced in the previous section. To determine the bounds on the family of constants $\{C^m_{n,k}\}$ in Theorem 2.1 we will make use of the Fourier coefficients of Fučík eigenfunctions corresponding to Fučík eigenvalues on the first nontrivial curve Γ_2. Namely, we provide estimates for the Fourier coefficients of the odd Fourier expansion of the function

$$g^2_{\gamma,\gamma/(\sqrt{\gamma}-1)^2} = \sum_{k=1}^{\infty} A_k(\gamma)\varphi_k(x)$$

for $\gamma > 4$ which are given by

$$A_k(\gamma) = \int_0^{\pi} g^2_{\gamma,\gamma/(\sqrt{\gamma}-1)^2}(x)\varphi_k(x)\,\mathrm{d}x = \frac{2}{\pi}\frac{\gamma^2}{\sqrt{\gamma}-1}\frac{(2-\sqrt{\gamma})\sin\left(\frac{k\pi}{\sqrt{\gamma}}\right)}{(k^2-\gamma)(k^2(\sqrt{\gamma}-1)^2-\gamma)},$$

and of the function

$$g^2_{\delta/(\sqrt{\delta}-1)^2,\delta} = \sum_{k=1}^{\infty} \widetilde{A}_k(\delta)\varphi_k(x)$$

for $\delta > 4$ which are given by

$$\widetilde{A}_k(\delta) = \int_0^{\pi} g^2_{\delta/(\sqrt{\delta}-1)^2,\delta}(x)\varphi_k(x)\,\mathrm{d}x = (-1)^k A_k(\delta).$$

In the case $\gamma = \delta = 4$, we have $A_2 = 1$ and $A_k = 0$ for any other $k \in \mathbb{N}$.

Obviously, we have

$$|A_1(\gamma)| = B_1(\gamma) := \frac{2}{\pi}\frac{\gamma^2}{\sqrt{\gamma}-1}\frac{(\sqrt{\gamma}-2)\sin\left(\frac{\pi}{\sqrt{\gamma}}\right)}{(\gamma-1)(2\sqrt{\gamma}-1)} \tag{3.1}$$

and it was shown in [2, Section 5] that

$$|A_2(\gamma) - 1| \le B_2(\gamma) := \frac{((3+\pi^2)\gamma + (9-2\pi^2)\sqrt{\gamma} - 6)(\sqrt{\gamma}-2)}{3(\sqrt{\gamma}-1)(\sqrt{\gamma}+2)(3\sqrt{\gamma}-2)}. \tag{3.2}$$

For $\gamma \in [4, 9)$, we clearly have

$$|A_3(\gamma)| = B_3(\gamma) := \frac{2}{\pi}\frac{\gamma^2}{\sqrt{\gamma}-1}\frac{(\sqrt{\gamma}-2)\Big(-\sin\left(\frac{3\pi}{\sqrt{\gamma}}\right)\Big)}{(9-\gamma)(2\sqrt{\gamma}-3)(4\sqrt{\gamma}-3)} \tag{3.3}$$

and for $k \ge 4$ we use the simple estimate

$$|A_k(\gamma)| \le B_k(\gamma) := \frac{2}{\pi}\frac{\gamma^2}{\sqrt{\gamma}-1}\frac{(\sqrt{\gamma}-2)}{(k^2-\gamma)((k-1)\sqrt{\gamma}-k)((k+1)\sqrt{\gamma}-k)}. \tag{3.4}$$

Evidently, the same bounds hold for $\widetilde{A}_k$. Numerical calculations with the exact coefficients show that the used estimates in (3.2) and (3.4) do not influence the results in a significant way.

Lemma 3.1. *Let $\gamma \in [4, 9)$ and $k \in \mathbb{N}$. Then B_k is strictly increasing.*

Proof. For simplicity, we introduce the change of variables $x = \sqrt{\gamma} \in [2, 3)$. The first derivative of $B_k(x^2)$ with $k \in \mathbb{N} \setminus \{1, 3\}$ is a rational function with a positive denominator and we can easily check that the numerator is positive, as well. Hence, $B_k(\gamma)$ with $k \in \mathbb{N} \setminus \{1, 3\}$ is strictly increasing for $\gamma \in [4, 9)$. The first derivative of $B_1(x^2)$ takes the form

$$\frac{2x^2(x-1)\cos\left(\frac{\pi}{x}\right)\left[x(2x^4-4x^3-x^2+15x-8)\tan\left(\frac{\pi}{x}\right) - \pi(2x^4-5x^3+5x-2)\right]}{\pi(x-1)^2(x^2-1)^2(2x-1)^2}.$$

Noting that $x(2x^4 - 4x^3 - x^2 + 15x - 8) > 0$ for $x \in [2, 3)$, we can use the simple lower bound $\tan\left(\frac{\pi}{x}\right) \ge \sqrt{3}$ to show that the expression in square brackets is positive.

Since all other terms in the derivative are also positive, we conclude that $B_1(\gamma)$ is strictly increasing for $\gamma \in [4,9)$.

Finally, the numerator of the first derivative of $B_3(x^2)$ is given by

$$\begin{aligned} &-2x^2\Big[x(10x^5+90x^4-765x^3+1872x^2-1863x+648)\sin\big(\frac{3\pi}{x}\big) \\ &+3\pi(8x^6-42x^5+7x^4+315x^3-693x^2+567x-162)\cos\big(\frac{3\pi}{x}\big)\Big], \end{aligned} \tag{3.5}$$

whereas the denominator is a positive polynomial. We have $\sin\big(\frac{3\pi}{x}\big) < 0$ and $\cos\big(\frac{3\pi}{x}\big) < 0$ for $x \in [2,3)$, and taking into account that

$$\begin{gathered} x(10x^5+90x^4-765x^3+1872x^2-1863x+648) < 0, \\ 3\pi(8x^6-42x^5+7x^4+315x^3-693x^2+567x-162) > 0, \end{gathered}$$

we employ the estimates

$$\sin\big(\frac{3\pi}{x}\big) < -\Big(\frac{3\pi}{x}-\pi\Big)+\frac{1}{6}\Big(\frac{3\pi}{x}-\pi\Big)^3 \quad \text{and} \quad \cos\Big(\frac{3\pi}{x}\Big) > -1.$$

As a result, the expression (3.5) is estimated from below by a polynomial which is positive for $x \in [2,3)$. Thus, $B_3(\gamma)$ is strictly increasing for $\gamma \in [4,9)$. □

Now we are ready to prove our main result.

Proof of Theorem 1.3. We apply Theorem 2.1, where we consider $X = L^2(0,\pi)$, the sequence $\{f_n\}$ is the Fučík system, which is bounded by definition, and the complete orthonormal set $\{\phi_n\}$ is given by $\{\varphi_n\}$. We set $M = 1$ and $N_1 = N$ and choose $N_* = \mathbb{N} \setminus N$ as assumed in Theorem 1.3. We define the linear operators $T_k^1 \colon L^2(0,\pi) \to L^2(0,\pi)$ as

$$T_k^1 g(x) = g^*\Big(\frac{kx}{2}\Big),$$

where

$$g^*(x) = (-1)^\kappa g(x-\pi\kappa) \quad \text{for } \pi\kappa \le x \le \pi(\kappa+1), \quad \kappa \in \mathbb{N}\cup\{0\},$$

is the 2π-antiperiodic extension for arbitrary functions $g \in L^2(0,\pi)$. In particular, we have $T_k^1 \sin(nx) = \sin\big(\frac{knx}{2}\big)$ for every even n. It was proven in [2, Appendix B] that $\|T_k^1\|_* = 1$ for even k and $\|T_k^1\|_* = \sqrt{1+1/k}$ for odd k.

Let $n \in N$ be fixed and recall that n is even. To begin with, we assume that $\alpha(n) > n^2$. The Fučík eigenfunction $g^n_{\alpha,\beta}$ has the dilated structure

$$g^n_{\alpha,\beta}(x) = g^2_{\gamma_n,\gamma_n/(\sqrt{\gamma_n}-1)^2}\Big(\frac{nx}{2}\Big) \quad \text{with} \quad \gamma_n = \frac{4\alpha(n)}{n^2}$$

and, thus, has the odd Fourier expansion

$$g^n_{\alpha,\beta}(x) = g^2_{\gamma_n,\gamma_n/(\sqrt{\gamma_n}-1)^2}\Big(\frac{nx}{2}\Big) = \sum_{k=1}^\infty A_k(\gamma_n)\varphi_k\Big(\frac{nx}{2}\Big) = \sum_{k=1}^\infty A_k(\gamma_n) T_k^1\varphi_n(x).$$

From this, we directly see that the representation (2.1) of $g^n_{\alpha,\beta}$ in terms of $\{\varphi_n\}$ holds with the constants $C^1_{n,k} = A_k(\gamma_n)$ for $k \ne 2$ and $C^1_{n,2} = 1 - A_2(\gamma_n)$. The bounds for the constants $|C^1_{n,k}|$ are given by the functions $B_k(\gamma_n)$ defined in (3.1), (3.2), (3.3), and (3.4), which are strictly increasing in the interval $[4,9)$ by Lemma 3.1. For the case $\beta(n) > n^2$, the Fučík eigenfunction has the form

$$g^n_{\alpha,\beta}(x) = g^2_{\delta_n/(\sqrt{\delta_n}-1)^2,\delta_n}\Big(\frac{nx}{2}\Big) \quad \text{with} \quad \delta_n = \frac{4\beta(n)}{n^2},$$

and by analogous arguments we get the bounds $|C^1_{n,k}| \le B_k(\delta_n)$. If $\alpha(n) = n^2$, and hence $\beta(n) = n^2$, then we set $C^1_{n,k} = 0$ for every $k \in \mathbb{N}$.

In view of the monotonicity, we have

$$|C^1_{n,k}| \le B_k\Big(\sup_{n\in N} \max(\gamma_n, \delta_n)\Big).$$

Therefore, we can provide the following upper estimate on the constant Λ_1 defined in (2.2):

$$\begin{aligned}
\Lambda_1 &\le \sqrt{2}B_1\Big(\sup_{n\in N}\max(\gamma_n,\delta_n)\Big) + B_2\Big(\sup_{n\in N}\max(\gamma_n,\delta_n)\Big) \\
&\quad + \sqrt{\frac{4}{3}}B_3\Big(\sup_{n\in N}\max(\gamma_n,\delta_n)\Big) + B_4\Big(\sup_{n\in N}\max(\gamma_n,\delta_n)\Big) \\
&\quad + \sqrt{\frac{6}{5}}\sum_{k=5}^{\infty} B_k\Big(\sup_{n\in N}\max(\gamma_n,\delta_n)\Big) \\
&= E\Big(\sup_{n\in N}\max(\gamma_n,\delta_n)\Big) = E\Big(\sup_{n\in N}\Big\{\frac{4\max(\alpha(n),\beta(n))}{n^2}\Big\}\Big),
\end{aligned}$$

with the function E introduced in Theorem 1.3, and E is strictly increasing in $[4,9)$. Noticing that we have

$$\Lambda_* = \Big(\sum_{n\in N_*}\Big[1 - \frac{\langle g^n_{\alpha,\beta}, \varphi_n\rangle^2}{\|g^n_{\alpha,\beta}\|^2}\Big]\Big)^{1/2},$$

the assumption (1.2) yields the assumption $\Lambda_*^2 + \Lambda_1^2 < 1$ in Theorem 2.1. This completes the proof of Theorem 1.3. □

We conclude this note by discussing Lemma 1.4. The monotonicity statement (ii) directly follows from Lemma 3.1, and to obtain the alternative representation (iv), we make use of the identity

$$\sum_{k=1}^{\infty}\frac{1}{k^2 - a^2} = \frac{1}{2a^2} - \frac{\pi\cot(\pi a)}{2a}, \quad a \notin \mathbb{N},$$

see, e.g., [1, (6.3.13)]. The representation (iv) shows that the function E is continuous in $[4,9)$. The combination of the continuity and monotonicity of E allows us to compute values of E with an arbitrary precision. In particular, we have $E(6.49278\dots) = 1$.

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