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BILINEAR ESTIMATES POSED IN FINITE DOMAINS IN 2D AND 3D

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Abstract. In this article, we establish bilinear estimates for the nonlinear term appearing in fluid mechanics models posed in bounded domains of 2D and 3D. Also, we give an example to apply the estimates in a 2D bounded domain.

1. Introduction

We summarize and extend the bilinear estimate posed in a Sobolev space $Y_{\tau,T}$ for nonlinear term $(u \cdot \nabla)v$ which models convection effects in flow phenomena. Here (u, v) are velocity vectors in $\mathbb{R}^n$, $n = 2$ and 3 and they depend on time. Operator $u \cdot \nabla = \sum_{i=1}^n u_i \partial_{x_i}$, which models convection effect in fluids.

In this article, we extend the bilinear estimate by Bona, Sun and Zhang [3] on 1D finite domains to multi-dimensional finite domains. This technique has lots of applications in handling nonlinear terms when one uses semi-group mild solution formulation for nonlinear dispersive and dissipative equations [1, 2, 3, 4, 21]. For instance of one application is to formulate mild solution for the linear initial-value problem $y'(t) + Ay(t) = 0$, $y(0) = \phi$ as

$$y(t) = e^{-At}\phi.$$

Its nonlinear (bilinear) counterpart problem $y' + Ay + B(y, y) = 0$ has a mild solution form

$$y(t) = e^{-At}\phi + \int_0^t e^{-A(t-s)} B(y(s), y(s)) ds,$$

with the additional nonlinear term treated as a perturbation. If the former linear problem has been solved, then bilinear estimates developed can lead to the solvability of nonlinear one in same work function spaces such as H^s and/or $Y_{\tau,T}$ (defined in later context).

Via this fashion, one can even handle unbounded domains. Specifically, in [1, 2, 4], bilinear estimates are obtained in half-line spatial domain $\mathbb{R}_+$ with data prescribed on one bound ($x = 0$). More general development of this type of estimates and applications are in wellposedness theory in low-regularity spaces with

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negative indices for dispersive equations posed in unbounded domains. These related estimates are defined in Fourier restriction spaces such as Bourgain Spaces, and we would refer readers to content and references therein [11, 12] by Kenig et al, and the recent work [20] on coupled KdV equations by Yang and Zhang.

In light of the application facilitating existence theory of strong or weak solutions, Zhang et al including one of the authors considered KdV equations and dissipative PDEs, respectively on finite domains in 1D (see e.g. [17, 18, 19, 21]).

Similar mechanism to approach existence of solution and special case: forced oscillations in high dimensions could be pursued, which might be based on results in this article. For high dimension fluid models, such as 3D Navier-Stokes equations, results about forced oscillations could be found in series of works [13, 14, 9, 10]) by Serrin, Shinbrot, Kato et al, followed with [5, 7, 8] by Hsia et al from 1D Burgers to high dimensional models. They approached problem by using trilinear estimates and Galerkin's method (see Temam's famous monographs [15, 16]). From author's point of view, if using the bilinear estimates and the related techniques presented in [3, 17, 19], proof could be more compact and terse.

We would cite the bilinear estimate lemma for term $(uv)_x$ (including uv_x) in [3]:

Theorem 1.1. *There exists a constant C separated from T such that for any $T > 0$ and $(u, v) \in Y_{\tau,T}$,*

$$\int_{\tau}^{\tau+T} \|(u(s)v(s))_x\| ds \le C(T^{1/2} + T^{1/3}) \|u\|_{Y_{\tau,T}} \|v\|_{Y_{\tau,T}}. \tag{1.1}$$

where the norm $\|\cdot\|$ is of L^2, and the function space $Y_{\tau,T}$ is defined on spatial interval $I = [0, 1]$ as

$$Y_{t,T} = \{u \big| \sup_{t \le s \le t+T} \|u(s)\| + \Big(\int_t^{t+T} \|u_x(s)\|^2 ds\Big)^{1/2} < \infty\}$$

with norm

$$\|u\|_{Y_{t,T}} = \sup_{t \le s \le t+T} \|u(s)\| + \Big(\int_t^{t+T} \|u_x(s)\|^2 ds\Big)^{1/2}.$$

In this theorem, the function space $Y_{\tau,T}$ is L^2 adding a smoothing over time interval $[\tau, \tau + T]$. Zhang and one of the authors invoked this version in analyzing forced oscillation and its stability of 1D Burgers equation (see [19]). However, when dimension increases, Sobolev imbedding differs from case in $n = 1$, and we need variants for analysis on nonlinear term. We would mimic similar fashion but shift to higher regularity in H^1 to define suitable function spaces for 2D and 3D and seek their linear estimates. We have applied some of these estimates in some fluid models in high dimensions (see [6]).

This article is organized as follows: we present main results in Section 2, and their proofs in Section 3. We show the usage of the inequalities in Section 4.

2. Main results

Norm $\|\cdot\|_X$ is used to denote the endowed canonical norm of a Banach space X, and $\|\cdot\|$ denotes the L^2 norm. Since we treat u as a vector in $\mathbb{R}^n$, the L^2 norm of u is

$$\|u(t)\| = \Big(\int_\Omega \sum_{i=1}^n u_i^2(x,t) dx\Big)^{1/2} \quad \text{with } u = (u_1, u_2, \dots, u_n)$$

on a bounded domain Ω in $\mathbb{R}^n$. Also, given the context of flow following time evolution, we may use $u(t)$ or $\|u(t)\|_X$.

We will consider bilinear estimates related to this Sobolev function space

$$Y_{t,T} = \{u| \sup_{t\le s\le t+T} \|\nabla u(s)\| + \Big(\int_t^{t+T} \|Au(s)\|^2 ds\Big)^{1/2} < \infty\}$$

endowed with the norm

$$\|u\|_{Y_{t,T}} = \sup_{t\le s\le t+T} \|\nabla u(s)\| + \Big(\int_t^{t+T} \|Au(s)\|^2 ds\Big)^{1/2}$$

on the bounded time interval $[t, t+T]$.

Note that $Y_{t,T}$ is related to Hilbert space L^2 or H^1 relying on dimension n. ∇u is the Jacobian. Operator A is the Stokes operator which relates to second derivatives according to [15, 16] by Laplacian $\triangle$ and Leray projection operator $\mathbb{P}$:

$$Au = -\mathbb{P}\triangle u : H^2 \mapsto L^2, \text{ in the solenoidal vector field } \nabla\cdot u = 0,$$

which gives Au being a vector in $\mathbb{R}^n$.

Our results for convection nonlinear terms are presented as follows:

Theorem 2.1. *If $n = 2$ and 3, for given positive T and τ, there exists a positive number $C(T)$ such as*

$$\int_\tau^{\tau+T} \|u(s)\cdot\nabla v(s)\|_{H^1} ds \le C(T)\|u\|_{Y_{\tau,T}}\|v\|_{Y_{\tau,T}}. \tag{2.1}$$

We shall prove this result sequentially in 2D and 3D, when Poicaré inequality holds between the L^2 norms of ∇u and u, Au and ∇u.

3. Proof of main results

Proof of Theorem 2.1 when $n = 2$. We prove when $\tau = 0$. We first prove estimate on $\|u\cdot\nabla v\|$.

In 2D finite domains,

$$\|u\cdot\nabla v\| \le \|u\|_{L^4}\|\nabla v\|_{L^4} \le C\|\nabla u\|\|\nabla v\|_{L^4}.$$

From Hölder inequality and the Sobolev inequality, we have $\|u\|_{L^4} \lesssim \|\nabla u\|$. Owing to the Gagliardo-Nirenburg inequality in 2D finite domains,

$$\|\nabla v\|_{L^4} \lesssim \|\nabla v\| + \|\nabla v\|^{1/2}\|Av\|^{1/2},$$

we have

$$\int_0^T \|u(s)\cdot\nabla v(s)\| ds \le C\int_0^T \|\nabla u(s)\|(\|\nabla v(s)\| + \|\nabla v(s)\|^{1/2}\|Av(s)\|^{1/2}) ds$$

whence

$$\int_0^T \|u(s)\cdot\nabla v(s)\| ds \le C\|u\|_{Y_{0,T}}\|v\|_{Y_{0,T}},$$

in that

$$\begin{aligned}\int_0^T \|\nabla u(s)\|\|\nabla v(s)\| ds &\le \sup_{s\in[0,T]} \|\nabla v\| T^{1/2}\Big(\int_0^T \|\nabla u(s)\|^2 ds\Big)^{1/2}\\ &\le T^{1/2}\|v\|_{Y_{0,T}}\|u\|_{Y_{0,T}},\end{aligned}$$

$$
\begin{aligned}
&\int_0^T \|\nabla v(s)\|^{1/2}\|Av(s)\|^{1/2}\|\nabla u(s)\|ds \\
&\leq \sup_{s\in[0,T]}(\|\nabla u(s)\|\|\nabla v(s)\|)T^{3/4}\Big(\int_0^T \|Av(s)\|^2 ds\Big)^{1/4} \\
&\leq T^{3/4}\|u\|_{Y_{0,T}}\|v\|_{Y_{0,T}}.
\end{aligned}
$$

Second, on $\int_0^T \|\nabla(u(s)\cdot\nabla v(s))\|ds$, we have

$$
\begin{aligned}
&\int_0^T \|\nabla(u(s)\cdot\nabla v(s))\|ds \\
&\leq \int_0^T \|\nabla u(s)\cdot\nabla v(s)\|ds + \int_0^T \|u(s)\cdot Av(s)\|ds \\
&\leq \int_0^T \|\nabla u(s)\|_{L^4}\|\nabla v(s)\|_{L^4}ds + \int_0^T \|u(s)\|_{L^\infty}\|Av(s)\|ds \\
&\leq C\int_0^T (\|\nabla u(s)\| + \|\nabla u(s)\|^{1/2}\|Au(s)\|^{1/2})(\|\nabla v(s)\| + \|\nabla v(s)\|^{1/2}\|Av(s)\|^{1/2})ds \\
&\quad + C\int_0^T \|Au(s)\|\|Av(s)\|ds \\
&\leq CT^{3/4}\|u\|_{Y_{0,T}}\|v\|_{Y_{0,T}} + \sup_{s\in[0,T]}(\|\nabla u(s)\|\|\nabla v(s)\|)^{1/2} \\
&\quad \times T^{1/2}\Big(\int_0^T \|Au(s)\|^2 ds\Big)^{1/4}\Big(\int_0^T \|Av(s)\|^2 ds\Big)^{1/4} \\
&\leq C(T^{3/4} + T^{1/2} + 1)\|u\|_{Y_{0,T}}\|v\|_{Y_{0,T}}
\end{aligned}
$$

in which we use Hölder inequality, Poicaré inequality, Gagliardo-Nirenburg inequality, Sobolev inequality $\|u\|_{L^\infty} \lesssim \|Au\|$ in 2D finite domains. Combining these two estimates, we obtain the bilinear estimate (2.1) for 2D. □

Proof of Theorem 2.1 when $n = 3$. With a similar manner in last proof, we consider two terms:

$$
\int_0^T \|u\cdot\nabla v\|ds \quad \text{and} \quad \int_0^T \|\nabla(u\cdot\nabla v)\|ds.
$$

In 3D finite domains, one can invoke Hölder inequality to reach

$$
\begin{aligned}
\int_0^T \|u(s)\cdot\nabla v(s)\|ds &\leq C\int_0^T \|u(s)\|_{L^{12}}\|\nabla v(s)\|_{L^{12/5}}ds \\
&\leq C\int_0^T \|\nabla u(s)\|_{L^{12/5}}\|\nabla v(s)\|_{L^{12/5}}ds.
\end{aligned}
$$

with Sobolev inequality $\|u\|_{L^{12}} \lesssim \|\nabla u\|_{L^{12/5}}$ for $n = 3$.

Owing to the interpolation in L^p for ∇u,

$$
\|\nabla u\|_{L^{12/5}} \leq C\|\nabla u\|^{3/4}\|\nabla u\|_{L^6}^{1/4},
$$

and Sobolev inequality $\|\nabla u\|_{L^6} \lesssim \|Au\|$ for $n = 3$, whence

$$
\int_0^T \|u(s)\cdot\nabla v(s)\|ds
$$

$$\begin{aligned}
&\le C\Big(\int_0^T \|\nabla u(s)\|_{L^{12/5}}^2 ds\Big)^{1/2}\Big(\int_0^T \|\nabla v(s)\|_{L^{12/5}}^2 ds\Big)^{1/2}\\
&\le C\Big(\int_0^T \|\nabla u(s)\|^{3/2}\|\nabla u(s)\|_{L^6}^{1/2} ds\Big)^{1/2}\Big(\int_0^T \|\nabla v(s)\|^{3/2}\|\nabla v(s)\|_{L^6}^{1/2} ds\Big)^{1/2}\\
&\le C\Big(\int_0^T \|Au(s)\|^2 ds\Big)^{1/2}\Big(\int_0^T \|Av(s)\|^2 ds\Big)^{1/2}\\
&\le C\|u\|_{Y_{0,T}}\|v\|_{Y_{0,T}}.
\end{aligned}$$

For the second term, since $\|\nabla(u\cdot\nabla v)\| \lesssim \|\nabla u\cdot\nabla v\| + \|uAv\|$, we have

$$\begin{aligned}
\int_0^T \|\nabla u(s)\cdot\nabla v(s)\| ds &\lesssim \int_0^T \|\nabla u(s)\|_{L^4}\|\nabla v(s)\|_{L^4} ds\\
&\lesssim \int_0^T \|\nabla u(s)\|_{L^6}\|\nabla v(s)\|_{L^6} ds\\
&\lesssim \Big(\int_0^T \|Au(s)\|^2 ds\Big)^{1/2}\Big(\int_0^T \|Av(s)\|^2 ds\Big)^{1/2}\\
&\lesssim \|u\|_{Y_{0,T}}\|v\|_{Y_{0,T}},
\end{aligned}$$

by $\|\nabla u\|_{L^6} \lesssim \|Au\|$.

$$\begin{aligned}
\int_0^T \|u(s)Av(s)\| ds &\lesssim \int_0^T \|u(s)\|_{L^\infty}\|Av(s)\| ds\\
&\lesssim \int_0^T \|Au(s)\|\|Av(s)\| ds\\
&\lesssim \Big(\int_0^T \|Au(s)\|^2 ds\Big)^{1/2}\Big(\int_0^T \|Av(s)\|^2 ds\Big)^{1/2}\\
&\lesssim \|u\|_{Y_{0,T}}\|v\|_{Y_{0,T}}.
\end{aligned}$$

By Agmon's inequality, $\|u\|_{L^\infty} \lesssim \|u\|_{H^1}^{1/2}\|u\|_{H^2}^{1/2}$ when $n=3$. Therefore,

$$\int_0^T \|\nabla(u\cdot\nabla v)\| ds \le C\|u\|_{Y_{0,T}}\|v\|_{Y_{0,T}}.$$

Combining these two estimates for $\nabla(u\cdot\nabla v)$ and $u\cdot\nabla v$, we complete the proof. $\square$

4. Application

In this section, we show a usage of bilinear estimates by looking at a simplified model of 2D incompressible Navier-Stokes equation when external force f can be controlled. Some of the following practice has been recently used in [6] by some of authors.

Consider a Burgers-type equation posed in 2D finite domain $\Omega\times[\tau,\tau+T]$:

$$\frac{\partial}{\partial t}u + Au + u\cdot\nabla u = f \tag{4.1}$$

with homogeneous Dirichlet boundary condition (no-slip boundary condition). We also have initial value at $t=\tau$ that $u\in H^1(\Omega)$ (we will use H^1 for this space). We will obtain the following existence result of u.

Theorem 4.1. *If equation* (4.1) *with no-slip boundary conditions, H^1 initial value and $f \in H^1$, then*

$$u \in Y_{\tau,T}. \tag{4.2}$$

Let $u(x,t) = v(x,t) + z(x,t)$, where v satisfies linear equation

$$\begin{gathered} \frac{\partial}{\partial t} v + Av = f, \\ v(x,\tau) = u(x,\tau), \end{gathered} \tag{4.3}$$

and z satisfies the nonlinear equation

$$\begin{gathered} \frac{\partial}{\partial t} z + Az = -z \cdot \nabla z - z \cdot \nabla v - v \cdot \nabla z - v \cdot \nabla v, \\ z(x,\tau) = 0. \end{gathered} \tag{4.4}$$

We could use mild solution form when we take gradient on both sides of (4.1), to get the following $Y_{\tau,T}$-result for linear equation (4.3).

Lemma 4.2. *There exists a positive constant C not relying on T, such that*

$$\|v\|^2_{Y_{\tau,T}} \le C\Big[\|\nabla v(\tau)\|^2 + \Big(\int_\tau^{\tau+T} \|\nabla f(s)\|^2 ds\Big)\Big]. \tag{4.5}$$

We can reach a result for z when obtained magnitude of v ($\|v\|_{Y_{\tau,T}}$) is small.

Lemma 4.3. *If there exists $M > 0$ such that if $\|v\|_{Y_{\tau,T}} \le M$, then*

$$\|z\|_{Y_{\tau,T}} \le C\|v\|_{Y_{\tau,T}}. \tag{4.6}$$

Proof. We write a mild solution form for ∇z,

$$\nabla z(t) = -\int_\tau^t e^{\Delta(t-s)} \nabla \left(z \cdot \nabla z + z \cdot \nabla v + v \cdot \nabla z + v \cdot \nabla v\right) ds.S$$

for $t \in (\tau, \tau + T)$.

We define the map $q \mapsto \Gamma(q)$ referring to $v \in Y_{\tau,T}$: $Y_{\tau,T} \mapsto Y_{\tau,T}$ such that

$$\Gamma(q; v) = -\int_\tau^t e^{\Delta(t-s)} \nabla(q \cdot \nabla q + q \cdot \nabla v + v \cdot \nabla q + v \cdot \nabla v) ds.$$

We aim to prove that $\Gamma(q; v)$ has a fix point in space $Y_{\tau,T}$ to obtain the existence of $z(t)$.
We define a positive controlling constant M to bound the magnitude of v in $Y_{\tau,T}$ and a bounded set $S_M \in Y_{\tau,T}$:

$$M = \|v\|_{Y_{\tau,T}}, \quad S_M = \left\{q \in Y_{\tau,T} \middle| \|Q\|_{Y_{\tau,T}} \le M\right\}.$$

We will apply the Banach Fixed Point Theorem on S_M. We could prove $\Gamma(q) \in S_M$ if $q \in S_M$. In fact, by the bilinear estimate

$$\begin{aligned} \|\Gamma(q; v)\|_{Y_{0,T}} &\le \int_\tau^t \|(q \cdot \nabla q + q \cdot \nabla v + v \cdot \nabla q + v \cdot \nabla v\|_{H^1} ds \\ &\le \int_\tau^t (\|q\|^2_{Y_{\tau,T}} + \|q\|_{Y_{\tau,T}} \|v\|_{Y_{0,T}} + \|v\|^2_{Y_{\tau,T}}) ds \\ &\le C(\|q\|^2_{Y_{\tau,T}} + \|v\|^2_{Y_{\tau,T}}) \\ &\le CM^2, \end{aligned}$$

Let M be so small that $M \leq \frac{1}{2C}$, then we infer that

$$\|\Gamma(q;v)\|_{Y_{\tau,T}} \leq \frac{M}{2}.$$

At this point, we would prove the Lipschitz continuity of Γ with respect to q. Pick any two distinct $q_1,\ q_2 \in S_M$, then

$$\begin{aligned}
&\|\Gamma(q_1;v) - \Gamma(q_2;v)\|_{Y_{\tau,T}} \\
&= \int_\tau^t \|e^{\Delta(T-s)}\| \|\nabla(q_1 \cdot \nabla q_1 + q_1 \cdot \nabla v + v \cdot \nabla q_1 - q_2 \cdot \cdot q_1 - q_2 \cdot \nabla v - v \cdot \nabla q_2\| ds \\
&\leq C\|q_1 - q_2\|_{Y_{\tau,T}} \|q_1\|_{Y_{0,T}} + C\|q_1 - q_2\|_{Y_{\tau,T}} \|q_2\|_{Y_{0,T}} + C\|q_1 - q_2\|_{Y_{\tau,T}} \|v\|_{Y_{\tau,T}} \\
&\leq 3CM\|q_1 - q_2\|_{Y_{\tau,T}} \\
&\leq \frac{1}{2}\|q_1 - q_2\|_{Y_{\tau,T}}
\end{aligned}$$

if so small $M \leq \frac{1}{6C}$.

We take $M < 1/(6C)$, hence the above two estimates hold and the fix point is in S_M. □

Proof of Theorem 4.1. Combining Lemmas 4.2 and 4.3, we obtain the sum $u \in Y_{\tau,T}$. □

In the above analysis, we obtain the existence fo small data, i.e., when $u(\tau)$ and ∇f are sufficiently small in H^1.

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