2021/2023 UNC Greensboro PDE Conference,
Electronic Journal of Differential Equations, Conference 26 (2022/2025), pp. 201–217.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

NORMAL SOLVABILITY AND FREDHOLM PROPERTIES FOR SPECIAL CLASSES OF HYPOELLIPTIC OPERATORS

ANI TUMANYAN

ABSTRACT. In this work, we establish normal solvability and a priori estimates for hypoelliptic operators with special variable coefficients, associated with multi-quasi-elliptic symbols, acting in weighted Sobolev spaces in $\mathbb{R}^n$. We obtain Fredholm criteria for the special classes of regular hypoelliptic operators in various scales of multianisotropic spaces. We also provide applications to the smoothness of solutions, index invariance on the scale, and spectral properties of such operators.

1. INTRODUCTION, BASIC NOTIONS AND DEFINITIONS

We study the Fredholm properties of a class of regular hypoelliptic operators, which is a special subclass of Hörmander's hypoelliptic operators and has many important applications (see [16]). The characteristic polynomials of these operators are multi-quasi-elliptic, so they are a natural generalization of elliptic, parabolic, $2b$-parabolic, and quasielliptic operators. These operators were introduced in the late 60s-70s and studied by many authors: Nikolsky [22], Mikhailov [21], Friberg [13], Volevich, Gindikin [31], Ghazaryan [14], and others.

The analysis of regular hypoelliptic operators has certain challenges, as corresponding characteristic polynomials are not homogeneous like in the elliptic case. Solvability conditions, a priori estimates, and Fredholm properties have been studied for special classes of hypoelliptic operators in various functional spaces, but most of the results are related to elliptic and quasielliptic operators.

The Fredholm property for elliptic operators has been studied for different scales of weighted spaces in $\mathbb{R}^n$ in the works of Bagirov [3], Lockhart, McOwen [20, 19], Schrohe [24], and numerous others.

A priori estimates and the Fredholm solvability of quasielliptic operators have been studied in the works of Bagirov [4], Karapetyan, Darbinyan [17], Darbinyan and Tumanyan [8, 29], and others. Isomorphic characteristics for quasielliptic operators with constant coefficients on a special scale of weighted spaces have been derived in the works of Demidenko (see [10, 11]), and such operators have been

2020 *Mathematics Subject Classification.* 35H10, 35H30, 47A53.
Key words and phrases. Regular hypoelliptic operator; a priori estimate; Fredholm operator; multianisotropic weighted space.

Published May 13, 2025.

further studied in Hile's work (see [15]). In this paper, we obtain a priori estimates, normal solvability, and Fredholm criteria for a different class of quasielliptic operators with variable coefficients on the scale of weighted anisotropic spaces.

Rodino, Boggiatto, and Buzano studied the Fredholm properties and the spectrum of special classes of pseudo-differential operators in multianisotropic spaces with polynomial weights (see [5]). The spectral properties of Schrödinger type hypoelliptic operators, as well as hypoelliptic pseudo-differential operators, which are relatively bounded perturbations of constant-coefficients operators, have been studied in the works of Buzano, Ziggioto (see [6, 7]). Fredholm criteria have been established for specific subclasses of regular hypoelliptic operators in the works [26, 28].

In this article, we obtain normal solvability and a priori estimates for two classes of regular hypoelliptic operators with variable coefficients, acting on special scales of weighted Sobolev spaces in $\mathbb{R}^n$. Fredholm criteria are established for the considered classes of operators on multianisotropic $H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ and anisotropic $H_q^{k,\nu,p}(\mathbb{R}^n)$ scales of spaces with appropriate weight functions q. We study regularity of solutions, index invariance, and spectral properties of these operators. The scales of multianisotropic spaces and conditions on the coefficients considered are more general than those in previous works (see [29, 26]).

Definition 1.1. A bounded linear operator A, acting from a Banach space X to a Banach space Y, is called a normally solvable operator if the image of operator A is closed $(\operatorname{Im}(A) = \overline{\operatorname{Im}(A)})$.

An operator A is called an n-normally solvable (or n-normal) operator if it is normally solvable, and the kernel of operator A is finite-dimensional $(\dim\ker(A) < \infty)$.

An operator A is called a Fredholm operator if it is n-normal, and the cokernel of operator A is finite-dimensional $(\dim\operatorname{coker}(A) = \dim Y/\operatorname{Im}(A) < \infty)$.

Definition 1.2. For a closed operator A with a dense domain in a Banach space X, essential spectrum of A is the set $\sigma_{es}(A)$ of complex numbers λ such that $A-\lambda I$ is not a Fredholm operator.

The difference between the dimension of the kernel and the cokernel of operator A is called the index of the operator

$$\operatorname{ind}(A) = \dim\ker(A) - \dim\operatorname{coker}(A).$$

Definition 1.3. For a bounded linear operator A, acting from a Banach space X to a Banach space Y, the bounded linear operators $R_1 : Y \to X$ and $R_2 : Y \to X$ are called, respectively, left and right regularizers if the following holds: $R_1A = I_X + T_1, AR_2 = I_Y + T_2$, where I_X, I_Y are the identity operators, $T_1 : X \to X$ and $T_2 : Y \to Y$ are compact operators.

A bounded linear operator $R : Y \to X$ is called a regularizer for operator A if it is a left and right regularizer.

Let $n \in \mathbb{N}$ and $\mathbb{R}^n$ be the Euclidean n-dimensional space, $\mathbb{Z}_+^n$, $\mathbb{N}^n$ be the sets of n-dimensional multi-indices and multi-indices with natural components respectively. Let $\mathcal{N} \subset \mathbb{Z}_+^n$ be a finite set of multi-indices, $\mathcal{R} = \mathcal{R}(\mathcal{N})$ be a minimum convex polyhedron containing all the points $\mathcal{N}$.

Definition 1.4. A polyhedron $\mathcal{R}$ is called completely regular if the following holds: (a) $\mathcal{R}$ is a complete polyhedron: $\mathcal{R}$ has a vertex at the origin and further vertices on each coordinate axes in $\mathbb{R}^n$; (b) all components of the outer normals of $(n-1)$-dimensional non-coordinate faces of $\mathcal{R}$ are positive.

Let $\mathcal{R}$ be a completely regular polyhedron. Denote by $\mathcal{R}_j^{n-1}(j = 1, \dots, I_{n-1})$ $(n-1)$-dimensional non-coordinate faces of $\mathcal{R}$ with the corresponding outer normals μ^j such that all multi-indices $\alpha \in \mathcal{R}_j^{n-1}$ satisfy $(\alpha : \mu^j) = \frac{\alpha_1}{\mu_1^j} + \dots + \frac{\alpha_n}{\mu_n^j} = 1$, $\partial\mathcal{R} = \bigcup_{j=1}^{I_{n-1}} \mathcal{R}_j^{n-1}$. For $k > 0$ denote by $k\mathcal{R} := \{k\alpha = (k\alpha_1, k\alpha_2 \dots, k\alpha_n) : \alpha \in \mathcal{R}\}$.

Consider the differential operator

$$P(x, \mathbb{D}) = \sum_{\alpha \in \mathcal{R}} a_\alpha(x) D^\alpha, \tag{1.1}$$

where $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, $D_j = i^{-1} \frac{\partial}{\partial x_j}, x = (x_1, \dots, x_n) \in \mathbb{R}^n, a_\alpha(x) \in C(\mathbb{R}^n)$. Denote

$$P(x, \xi) = \sum_{\alpha \in \mathcal{R}} a_\alpha(x) \xi^\alpha. \tag{1.2}$$

For $\xi \in \mathbb{R}^n$ denote

$$|\xi|_{\mathcal{R}} = \sum_{\alpha \in \mathcal{R}} |\xi^\alpha|, |\xi|_{\partial\mathcal{R}} = \sum_{\alpha \in \partial\mathcal{R}} |\xi^\alpha|.$$

Definition 1.5. A differential operator $P(x, \mathbb{D})$ is called regular at a point $x_0 \in \mathbb{R}^n$, if there exists a constant $\delta > 0$ such that

$$1 + |P(x_0, \xi)| \geq \delta |\xi|_{\mathcal{R}}, \forall \xi \in \mathbb{R}^n.$$

$P(x, \mathbb{D})$ is called regular in $\mathbb{R}^n$, if $P(x, \mathbb{D})$ is regular at each point $x \in \mathbb{R}^n$.

$P(x, \mathbb{D})$ is called uniformly regular in $\mathbb{R}^n$, if there exists a constant $\delta > 0$ such that:

$$1 + |P(x, \xi)| \geq \delta |\xi|_{\mathcal{R}}, \quad \forall \xi \in \mathbb{R}^n, \forall x \in \mathbb{R}^n.$$

A polyhedron $\mathcal{R}$ is called the characteristic or Newton polyhedron of $P(x, \mathbb{D})$.

In the anisotropic case, a completely regular polyhedron $\mathcal{R}$ has only one $(n-1)$-dimensional non-coordinate face with an outer normal $\nu \in \mathbb{N}^n$. The differential operator $P(x, \mathbb{D})$ with such characteristic polyhedron $\mathcal{R}$ can be written as

$$P(x, \xi) = \sum_{(\alpha:\nu) \leq 1} a_\alpha(x) \xi^\alpha. \tag{1.3}$$

For $\xi \in \mathbb{R}^n$ and $\nu \in \mathbb{N}^n$, denote $|\xi|_\nu := \sum_{i=1}^n |\xi_i^{\nu_i}|$.

Example 1.6. We ahve the followfing examples of regular differential operators:

(1) Let $m \in \mathbb{N}$ and $\mathcal{R}$ be a Newton polyhedron for the set of points $(0, 0, \dots, 0)$, $(m, 0, \dots, 0), \dots, (0, 0, \dots, m)$. In this case conditions from definition 1.5 coincide with ellipticity conditions with $|\xi|_{\partial\mathcal{R}} = |\xi|_m = \sum_{i=1}^n |\xi_i^m|$.

(2) Let $\nu \in \mathbb{N}^n$ and $\mathcal{R}$ be a Newton polyhedron for the set of points $(0, 0, \dots, 0)$, $(\nu_1, 0, \dots, 0), \dots, (0, 0, \dots, \nu_n)$. In this case conditions from definition 1.5 coincide with quasiellipticity conditions with $|\xi|_{\partial\mathcal{R}} = |\xi|_\nu = \sum_{i=1}^n |\xi_i^{\nu_i}|$.

(3) Let $n = 2$ and $\mathcal{R}$ be a Newton polyhedron for the points $(0, 0)$, $(8, 0)$, $(0, 8)$ and $(6, 4)$. Then

$$P(x, \mathbb{D}) = a_1 D_1^8 + a_2 D_1^6 D_2^4 + a_3 D_2^8 + q(x)$$

is a regular differential operator in $\mathbb{R}^2$ with some $a_1, a_2, a_3 > 0$ and $q \in C(\mathbb{R}^2)$.

(4) Let $n = 3$ and $\mathcal{R}$ be a Newton polyhedron for the points $(0,0,0)$, $(8,0,0)$, $(0,8,0)$, $(6,4,0)$, $(6,0,6)$, $(0,6,6)$ and $(0,0,12)$. Then
$$P(x,\mathbb{D}) = D_1^8 + D_1^6 D_2^4 + D_2^8 + D_1^6 D_3^6 + D_2^6 D_3^6 + D_3^{12} + q(x)$$
is a regular differential operator in $\mathbb{R}^3$ with $q \in C(\mathbb{R}^3)$.

Let the sequence $\{a_i\}_{i=0}^{\infty} \subset \mathbb{R}_+$ be such that the series $\sum_{i=0}^{\infty} a_i$ diverges, and the inequality $a_{i+1} < \gamma a_i$ with $\gamma > 0$ holds for $i = 0, 1, \dots$.

Using the sequence $\{a_i\}_{i=0}^{\infty}$, we define the special covering of $\mathbb{R}^n$ as $\{W_p\}_{p=1}^{\infty}$ and the sets of functions $\{\varphi_p\}_{p=1}^{\infty}$ and $\{\psi_p\}_{p=1}^{\infty}$, following the definitions in the works [3] and [26].

These systems of functions have the following properties:

(1) $\operatorname{supp}\varphi_p \subset \operatorname{supp}\psi_p \subset W_p$;
(2) $\psi_p(x)\varphi_p(x) = \varphi_p(x)$ for all $x \in \mathbb{R}^n$;
(3) for each $\alpha \in \mathbb{Z}_+$, there exists a constant $C_\alpha > 0$ such that
$$|D^\alpha \psi_p(x)| \le C_\alpha (a_{[\frac{p-1}{l}]})^{-|\alpha|}, \quad |D^\alpha \varphi_p(x)| \le C_\alpha (a_{[\frac{p-1}{l}]})^{-|\alpha|},$$
for all $x \in \mathbb{R}^n$, $p = 1, 2, \dots$;
(4) $\sum_{p=1}^{\infty} \varphi_p(x) \equiv 1$.

We denote
$$Q := \{g \in C(\mathbb{R}^n) : g(x) > 0, \forall x \in \mathbb{R}^n\}.$$

Further, we define two sets of special weight functions. For $m \in \mathbb{Z}_+$ and a completely regular polyhedron $\mathcal{R}$, denote as $Q^{m,\mathcal{R}}$ a set of weight functions $g \in Q$, which satisfy the following conditions:

(1) $\frac{1}{g(x)} \rightrightarrows 0$ when $|x| \to \infty$;
(2) for $\beta \in m\mathcal{R}, \beta \ne 0$ $D^\beta g(x) \in C(\mathbb{R}^n)$ and there exists $C_\beta > 0$ such that $\frac{|D^\beta g(x)|}{g(x)^{1+(\beta:\mu^j)}} \le C_\beta$ for all $x \in \mathbb{R}^n$, $j = 1, \dots, I_{n-1}$;
(3) for each $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ and $p_0 = p_0(\varepsilon) > 0$ such that for all $p > p_0$ when $\max_{j=1,\dots,l} \operatorname{diam} U_j < \delta$ the following holds:
$$\max_{x,y \in \overline{W}_p} \frac{|g(x) - g(y)|}{g(y)} < \varepsilon, \quad \max_{x,y \in \overline{W}_p} \frac{1}{g(x)^{\frac{1}{\mu_{max}}} a_{[\frac{p-1}{l}]}} < \varepsilon,$$
where $\mu_{max} = \max_{1 \le i \le I_{n-1}} \max_{1 \le s \le n} \{\mu_s^i\}$.

The considered class $Q^{m,\mathcal{R}}$ includes polynomial functions and special exponential functions such as: $(1 + |x|_{\mathcal{R}})^l, \exp{(1 + |x|_{\mathcal{R}})^r}$ when $l, r > 0$.

For $m \in \mathbb{Z}_+$ and $\nu \in \mathbb{N}^n$, denote as $\widetilde{Q}^{k,\nu}$ a set of weight functions $g \in Q$, which satisfy the following conditions:

(1) there exists a constant $C > 0$ such that $0 < g(x) \le C$ for all $x \in \mathbb{R}^n$;
(2) for $\beta \in \mathbb{Z}_+^n, (\beta : \nu) \le m, \beta \ne 0$ $D^\beta g \in C(\mathbb{R}^n)$ and there exists $C_\beta > 0$ such that $\frac{|D^\beta g(x)|}{g(x)^{1+(\beta:\nu)}} \le C_\beta$ for all $x \in \mathbb{R}^n$;
(3) for each $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ and $p_0 = p_0(\varepsilon) > 0$ such that for all $p > p_0$ when $\max_{j=1,\dots,l} \operatorname{diam} U_j < \delta$ the following holds:
$$\max_{x,y \in \overline{W}_p} \frac{|g(x) - g(y)|}{g(y)} < \varepsilon, \quad \max_{x,y \in \overline{W}_p} \frac{1}{g(x)^{\frac{1}{\nu_{\min}}} a_{[\frac{p-1}{l}]}} < \varepsilon,$$
where $\nu_{\min} = \min_{1 \le i \le n} \{\nu_i\}$.

The class $\widetilde{Q}^{m,\nu}$ includes functions $(1+|x|_\nu)^l$, where $-\frac{\nu_{\min}}{\nu_{max}} < l \le 0$. For $k \in \mathbb{R}$, a completely regular polyhedron $\mathcal{R}$ and $1 < p < \infty$, we denote

$$H^{k,\mathcal{R},p}(\mathbb{R}^n) := \{u \in S' : \|u\|_{k,\mathcal{R},p} := \|F^{-1}(1+|\xi|_{\partial\mathcal{R}})^k F u\|_{L_p(\mathbb{R}^n)} < \infty\},$$

where S' is the set of tempered distributions.

For $\Omega \subset \mathbb{R}^n$, we denote by $\dot{H}^{k,\mathcal{R},p}(\Omega)$ the completeness of $C_0^\infty(\Omega)$ with the norm $\|\cdot\|_{k,\mathcal{R},p}$.

For $k \in \mathbb{Z}_+$, a completely regular polyhedron $\mathcal{R}$, $1 < p < \infty$ and $q \in Q$, we denote

$$\begin{aligned} H_q^{k,\mathcal{R},p}(\mathbb{R}^n) := \big\{u : \|u\|_{H_q^{k,\mathcal{R},p}(\mathbb{R}^n)} &:= \|u\|_{k,\mathcal{R},p,q} \\ &:= \sum_{\alpha \in k\mathcal{R}} \|D^\alpha u \cdot q^{k-\max_i(\alpha:\mu^i)}\|_{L_p(\mathbb{R}^n)} < \infty\big\}. \end{aligned}$$

For $\Omega \subset \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$, we denote

$$H_q^{k,\mathcal{R},p}(\Omega) := \big\{u : \|u\|_{H_q^{k,\mathcal{R},p}(\Omega)} := \sum_{\alpha \in k\mathcal{R}} \|D^\alpha u \cdot q^{k-\max_i(\alpha:\mu^i)}\|_{L_p(\Omega)} < \infty\big\},$$

$$\begin{aligned} H_{q(x_0)}^{k,\mathcal{R},p}(\mathbb{R}^n) := \big\{u : \|u\|_{H_{q(x_0)}^{k,\mathcal{R},p}(\mathbb{R}^n)} &:= \|u\|_{k,\mathcal{R},p,q(x_0)} \\ &:= \sum_{\alpha \in k\mathcal{R}} \|D^\alpha u \cdot q(x_0)^{k-\max_i(\alpha:\mu^i)}\|_{L_p(\mathbb{R}^n)} < \infty\big\}. \end{aligned}$$

For $p = 2$, we denote $H^{k,\mathcal{R}}(\mathbb{R}^n) := H^{k,\mathcal{R},2}(\mathbb{R}^n)$ and $H_q^{k,\mathcal{R}}(\mathbb{R}^n) := H_q^{k,\mathcal{R},2}(\mathbb{R}^n)$.

In the anisotropic case, when $\mathcal{R}$ has only one $(n-1)$-dimensional non-coordinate face with an outer normal $\nu \in \mathbb{N}^n$, we analogously define the space $H^{k,\nu,p}(\mathbb{R}^n)$ for $k \in \mathbb{R}$ and the spaces $H_q^{k,\nu,p}(\mathbb{R}^n)$, $H_q^{k,\nu,p}(\Omega)$, $H_{q(x_0)}^{k,\nu,p}(\mathbb{R}^n)$ for $k \in \mathbb{Z}_+$. The introduced spaces generalize multianisotropic Sobolev-type spaces (see [14]).

2. A priori estimates and normal solvability

Let $k \in \mathbb{Z}_+$ and $q \in Q$. Consider the differential operator $P(x,\mathbb{D})$ (see (1.1)) with the coefficients that satisfy the following conditions:

$$P(x,\mathbb{D}) = \sum_{\alpha \in \mathcal{R}} a_\alpha(x) D^\alpha = \sum_{\alpha \in \mathcal{R}} \Big(a_\alpha^0(x) q(x)^{1-\max_i(\alpha:\mu^i)} + a_\alpha^1(x)\Big) D^\alpha, \tag{2.1}$$

where $D^\beta(a_\alpha^0(x)) = O\big(q(x)^{\min_i(\beta:\mu^i)}\big)$ and $D^\beta(a_\alpha^1(x)) = o\big(q(x)^{1-\max_i(\alpha-\beta:\mu^i)}\big)$ when $|x| \to \infty$ for all $\alpha \in \mathcal{R}, \beta \in k\mathcal{R}$.

It is easy to check that $P(x,\mathbb{D})$ generates a bounded linear operator, acting from $H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n)$ to $H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$.

We consider also the special case, when $\mathcal{R}$ has only one $(n-1)$-dimensional non-coordinate face with an outer normal $\nu \in \mathbb{N}^n$. In this case, the differential operator $P(x,\mathbb{D})$ can be expressed as

$$P(x,\mathbb{D}) = \sum_{(\alpha:\nu)\le 1} a_\alpha(x) D^\alpha = \sum_{(\alpha:\nu)\le 1} \Big(a_\alpha^0(x) q(x)^{1-(\alpha:\nu)} + a_\alpha^1(x)\Big) D^\alpha, \tag{2.2}$$

where $D^\beta(a_\alpha^0(x)) = O(q(x)^{(\beta:\nu)})$, $D^\beta(a_\alpha^1(x)) = o(q(x)^{1-(\alpha-\beta:\nu)})$ when $|x| \to \infty$ for all $(\alpha:\nu) \le 1, (\beta:\nu) \le k$. For $N > 0$ and $x_0 \in \mathbb{R}^n$ denote

$$K_N(x_0) := \{x \in \mathbb{R}^n : |x - x_0| \le N\}, K_N := K_N(0).$$

Further, we use the following theorem, a consequence of [18, Theorem 7.1].

Theorem 2.1. *Let $k \in \mathbb{Z}_+, q \in Q$, and $P(x,\mathbb{D})$ be the differential operator* (2.1). *Then the operator $P(x,\mathbb{D}) : H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ is an $n-$normal operator if and only if there exist constants $\kappa > 0$ and $N > 0$ such that*

$$\|u\|_{k+1,\mathcal{R},p,q} \leq \kappa(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_N)}), \quad \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n). \tag{2.3}$$

Further, we consider two classes of regular hypoelliptic operators defined using the weight functions from $Q^{k,\mathcal{R}}$ and $\widetilde{Q}^{k,\nu}$, respectively. We derive special conditions on the symbol of operators for the fulfillment of a priori estimates (2.3) for these special classes of regular hypoelliptic operators.

Theorem 2.2. *Let $k \in \mathbb{Z}_+, q \in Q^{k,\mathcal{R}}$, and $P(x,\mathbb{D})$ be the differential operator given by* (2.1) *with the coefficients satisfying $\lim_{m\to\infty} \max_{x,y\in\overline{W_m}} |a_\alpha^0(x) - a_\alpha^0(y)| = 0$ for all $\alpha \in \mathcal{R}$. Suppose there exists a constant $\kappa > 0$ such that*

$$\|u\|_{k+1,\mathcal{R},p,q} \leq \kappa(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(\mathbb{R}^n)}), \quad \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n). \tag{2.4}$$

Then, $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$, and there exist constants $\delta > 0$ and $M > 0$ such that

$$\Big|\sum_{\alpha\in\mathcal{R}} a_\alpha^0(x)\xi^\alpha\Big| \geq \delta(1 + |\xi|_{\partial\mathcal{R}}), \quad \forall \xi \in \mathbb{R}^n, |x| > M. \tag{2.5}$$

Proof. The proof of [26, Theorem 2.2] for $p = 2$ can be generalized for the spaces $H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n)$ in a similar way. $\square$

Theorem 2.3. *Let $k \in \mathbb{Z}_+, q \in \widetilde{Q}^{k,\nu}$, and $P(x,\mathbb{D})$ be the differential operator given by* (2.2) *with the coefficients satisfying $\lim_{m\to\infty} \max_{x,y\in\overline{W_m}} |a_\alpha^0(x) - a_\alpha^0(y)| = 0$ for all $\alpha \in \mathbb{Z}_+^n, (\alpha : \nu) \leq 1$. Suppose there exist constants $\kappa > 0$ and $N > 0$ such that:*

$$\|u\|_{k+1,\nu,p,q} \leq \kappa(\|Pu\|_{k,\nu,p,q} + \|u\|_{L_p(K_N)}), \forall u \in H_q^{k+1,\nu,p}(\mathbb{R}^n). \tag{2.6}$$

Then, $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$, and there exist constants $\delta > 0$ and $M > 0$ such that

$$\Big|\sum_{(\alpha:\nu)\leq 1} a_\alpha^0(x)\xi^\alpha\Big| \geq \delta(1 + |\xi|_\nu), \forall \xi \in \mathbb{R}^n, |x| > M. \tag{2.7}$$

Proof. From [29, Theorem 2.1] follows that $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$. Thus, we need to prove (2.7).

Let $\{x_m\}_{m=1}^\infty \subset \mathbb{R}^n$ is such a sequence that $|x_m| \to \infty$ when $m \to \infty$. Without loss of generality assume $x_m \in W_m$. Let $\xi \in \mathbb{R}^n$. Consider the function $\tilde{u}_m(x) = \exp(i(q(x_m)^{\frac{1}{\nu}}\xi, x))\psi_m(x)$. Since $q \in \widetilde{Q}^{k,\nu}$, then for any $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ and $m_0(\varepsilon) > 0$ such that for all $m > m_0$ and $\max_{j=1,\dots,l} \operatorname{diam} U_j < \delta$

$$|q(x) - q(y)| \leq \varepsilon q(y), \forall x, y \in W_m.$$

Then, for any $r > 0$ it holds

$$|q(x)^r - q(x_m)^r| \leq \tau_r(\varepsilon) q(x_m)^r, \forall x \in W_m, \tag{2.8}$$

where $\tau_r(\varepsilon) \to 0$ when $\varepsilon \to 0$.

From inequality (2.8) and $\operatorname{supp} \widetilde{u}_m \subset W_m$ follows that there exists $\tau(\varepsilon)$ such that $\tau(\varepsilon) \to 0$ when $\varepsilon \to 0$ and the following inequalities hold:

$$\|\tilde{u}_m\|_{k+1,\nu,p,q} \geq (1 - \tau(\varepsilon))\|\tilde{u}_m\|_{k+1,\nu,p,q(x_m)}, \tag{2.9}$$

$$\|P\tilde{u}_m\|_{k,\nu,p,q} \leq (1 + \tau(\varepsilon))\|P\tilde{u}_m\|_{k,\nu,p,q(x_m)}. \tag{2.10}$$

For sufficiently large $m_0 \in \mathbb{N}$ and a small enough $\max_{j=0,\dots,l} \operatorname{diam} U_j$ for $m > m_0$, it holds

$$\|\tilde{u}_m\|_{k+1,\nu,p,q} \geq \frac{1}{2}\|\tilde{u}_m\|_{k+1,\nu,p,q(x_m)}, \tag{2.11}$$

$$\|P\tilde{u}_m\|_{k,\nu,p,q} \leq \frac{1}{2}\|P\tilde{u}_m\|_{k,\nu,p,q(x_m)}. \tag{2.12}$$

Considering that $q \in \widetilde{Q}^{k,\nu}$ and the properties of $\{\psi_m\}_{m=1}^{\infty}$, we obtain that for all $(\gamma:\nu) \leq k+1$ and $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ and $m_0(\varepsilon) > 0$ such that for $m > m_0$ and $\max_{j=1,\dots,l} \operatorname{diam} U_j < \delta$, the following inequality holds

$$\frac{|D^\gamma \psi_m(x)|}{q(x)^{(\gamma:\nu)}} = \frac{|D^\gamma \psi_m(x)| a^{|\gamma|}_{[\frac{m-1}{l}]}}{q(x)^{(\gamma:\nu)-\frac{|\gamma|}{\nu_{\min}}} q(x)^{\frac{|\gamma|}{\nu_{\min}}} a^{|\gamma|}_{[\frac{m-1}{l}]}} \leq \omega_\gamma(\varepsilon), \tag{2.13}$$

where $\omega_\gamma(\varepsilon) \to 0$ when $\varepsilon \to 0$.

For $\beta \in \mathbb{Z}_+^n$ with some constants $C_1 > 0$ and $\sigma = \sigma(\nu) > 0$ the following holds:

$$\begin{aligned}
&\|D^\beta \tilde{u}_m\|_{L_p(\mathbb{R}^n)} q(x_m)^{k+1-(\beta:\nu)} \\
&\geq |\xi^\beta| q(x_m)^{k+1} \|\psi_m\|_{L_p(\mathbb{R}^n)} - C_1 \sum_{0 \leq \gamma < \beta} |\xi^\gamma| q(x_m)^{k+1-(\beta-\gamma:\nu)} \|D^{\beta-\gamma}\psi_m\|_{L_p(\mathbb{R}^n)}.
\end{aligned}$$

Using estimate (2.13) and the properties of $\{\psi_m\}_{m=1}^{\infty}$, the following estimate holds

$$\begin{aligned}
&\|D^\beta \tilde{u}_m\|_{L_p(\mathbb{R}^n)} q(x_m)^{k+1-(\beta:\nu)} \\
&\geq |\xi^\beta| q(x_m)^{k+1} \mu(W_m) - \omega_1(\varepsilon) \sum_{0 \leq \gamma < \beta} |\xi^\gamma| q(x_m)^{k+1} \mu(W_m),
\end{aligned}$$

where $\mu(W_m)$ is the measure of W_m, $\omega_1(\varepsilon) \to 0$ when $\varepsilon \to 0$.

Then, it is easy to check that the following holds:

$$\begin{aligned}
&\|\tilde{u}_m\|_{k+1,\nu,p,q(x_m)} \\
&\geq \sum_{(\beta:\nu) \leq k+1} |\xi^\beta| q(x_m)^{k+1} \mu_(W_m) - \omega_2(\varepsilon) \sum_{(\gamma:\nu) \leq k+1} |\xi^\gamma| q(x_m)^{k+1} \mu(W_m),
\end{aligned} \tag{2.14}$$

where $\omega_2(\varepsilon) \to 0$ when $\varepsilon \to 0$.

For $\beta \in \mathbb{Z}_+^n (\beta:\nu) \leq k$, we have

$$\begin{aligned}
&\|D^\beta (P(x,\mathbb{D})\tilde{u}_m)\|_{L_p(\mathbb{R}^n)} q(x_m)^{k-(\beta:\nu)} \\
&\leq \big\| D^\beta \Big(\sum_{(\alpha:\nu) \leq 1} a^0_\alpha(x) q(x)^{1-(\alpha:\nu)} D^\alpha \tilde{u}_m \Big) \big\|_{L_p(\mathbb{R}^n)} q(x_m)^{k-(\beta:\nu)} \\
&\quad + \big\| D^\beta (\sum_{\alpha \in \mathcal{R}} a^1_\alpha(x) D^\alpha \tilde{u}_m) \big\|_{L_p(\mathbb{R}^n)} q(x_m)^{k-(\beta:\nu)}.
\end{aligned} \tag{2.15}$$

Taking into account that $D^\beta(a^1_\alpha(x)) = o(q(x)^{1-(\alpha-\beta:\nu)})$ when $|x| \to \infty$ for all $\alpha, \beta \in \mathbb{Z}_+^n, (\alpha:\nu) \leq 1, (\beta:\nu) \leq k$ and (2.13), it is easy to check that for a sufficiently large m_0 and $m > m_0$, it holds

$$\begin{aligned}
&\big\| D^\beta (\sum_{\alpha \in \mathcal{R}} a^1_\alpha(x) D^\alpha \tilde{u}_m) \big\|_{L_p(\mathbb{R}^n)} q(x_m)^{k-(\beta:\nu)} \\
&\leq \omega_3(\varepsilon) \sum_{(\gamma:\nu) \leq k+1} |\xi^\gamma| q(x_m)^{k+1} \mu(W_m),
\end{aligned} \tag{2.16}$$

where $\omega_3(\varepsilon) \to 0$ when $\varepsilon \to 0$. From conditions (2.1), $\lim_{m\to\infty} \max_{x,y\in\overline{W_m}} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for all $(\alpha:\nu) \le 1$, $q \in \widetilde{Q}^{k,\nu}$, and inequality (2.8), we conclude that for $(\alpha:\nu) \le 1$ and $(\beta:\nu) \le k$ when m_0 is large enough and $\max_{j=1,\dots,l} \operatorname{diam} U_j$ is small enough, for $m > m_0$, the following holds:

$$|D^\beta(a^0_\alpha(x)q(x)^{1-(\alpha:\nu)} - a^0_\alpha(x_m)q(x_m)^{1-(\alpha:\nu)})| \le \tau_{\alpha,\beta}(\varepsilon)q(x_m)^{1-(\alpha:\nu)+(\beta:\nu)}, \tag{2.17}$$

where $\tau_{\alpha,\beta}(\varepsilon) \to 0$ when $\varepsilon \to 0$. Utilizing (2.17) and following a similar approach to the proof in [27, Theorem 2.4], we obtain that for a sufficiently large m_0 and $m > m_0$, the following estimate holds:

$$\begin{aligned}
&\Big\|D^\beta\Big(\sum_{(\alpha:\nu)\le 1} a^0_\alpha(x)q(x)^{1-(\alpha:\nu)}D^\alpha\tilde{u}_m\Big)\Big\|_{L_p(\mathbb{R}^n)}q(x_m)^{k-(\beta:\nu)}\\
&\le \Big|\sum_{(\alpha:\nu)\le 1} a^0_\alpha(x_m)\xi^\alpha\Big||\xi^\beta|q(x_m)^{k+1}\|\psi_m\|_{L_p(\mathbb{R}^n)}\\
&\quad + C_2\sum_{0\le\gamma<\beta}|\xi^\gamma|q(x_m)^{k+1-(\beta-\gamma:\nu)}\|D^{\beta-\gamma}\psi_m\|_{L_p(\mathbb{R}^n)}\\
&\le \Big|\sum_{(\alpha:\nu)\le 1} a^0_\alpha(x_m)\xi^\alpha\Big||\xi^\beta|q(x_m)^{k+1}\mu(W_m) + \omega_4(\varepsilon)\sum_{0\le\gamma<\beta}|\xi^\gamma|q(x_m)^{k+1}\mu(W_m)
\end{aligned} \tag{2.18}$$

where $\omega_4(\varepsilon) \to 0$ when $\varepsilon \to 0$.

From estimates (2.16)–(2.18) for a sufficiently large m_0, for all $m > m_0$, we obtain

$$\begin{aligned}
\|P\tilde{u}_m\|_{k,\nu,p,q(x_m)} &\le \Big|\sum_{(\alpha:\nu)\le 1} a^0_\alpha(x_m)\xi^\alpha\Big|\sum_{(\beta:\nu)\le k}|\xi^\beta|q(x_m)^{k+1}\mu(W_m)\\
&\quad + \omega_5(\varepsilon)\sum_{(\gamma:\nu)\le k+1}|\xi^\gamma|q(x_m)^{k+1}\mu(W_m),
\end{aligned} \tag{2.19}$$

where $\omega_5(\varepsilon) \to 0$ when $\varepsilon \to 0$. Then, from (2.4), using (2.14) and (2.19), we obtain

$$\begin{aligned}
&\sum_{(\beta:\nu)\le k+1}|\xi^\beta|q(x_m)^{k+1}\mu(W_m) - \omega_2(\varepsilon)\sum_{(\gamma:\nu)<k+1}|\xi^\gamma|q(x_m)^{k+1}\mu(W_m)\\
&\le \kappa\Big(\Big|\sum_{(\alpha:\nu)\le 1} a^0_\alpha(x_m)\xi^\alpha\Big|\sum_{(\beta:\nu)\le k}|\xi^\beta|q(x_m)^{k+1}\mu(W_m)\\
&\quad + \omega_5(\varepsilon)\sum_{(\gamma:\nu)\le k+1}|\xi^\gamma|q(x_m)^{k+1}\mu(W_m)\Big).
\end{aligned}$$

Since $\{a^0_\alpha(x) : (\alpha:\nu) \le 1\}$ are bounded functions and $x_m \to \infty$ when $m \to \infty$, there exist convergent subsequences of sequences $\{a^0_\alpha(x_m) : (\alpha:\nu) \le 1\}$. Without loss of generality, assume that the sequences $\{a^0_\alpha(x_m) : (\alpha:\nu) \le 1\}$ are convergent. So, for each $\alpha \in \mathbb{Z}^n_+, (\alpha:\nu) \le 1$ there exists a constant $\tilde{a}^0_\alpha$ such that $a^0_\alpha(x_m) \rightrightarrows \tilde{a}^0_\alpha$ when $m \to \infty$. Then, dividing by $q(x_m)^{k+1}\mu(W_m)$ and choosing m_0 sufficiently large, for $m > m_0$ we obtain the following inequality:

$$\sum_{(\beta:\nu)\le k+1}|\xi^\beta| - \omega_6(\varepsilon)\sum_{(\gamma:\nu)\le k+1}|\xi^\gamma| \le \kappa\Big|\sum_{(\alpha:\nu)\le 1}\tilde{a}^0_\alpha\xi^\alpha\Big|\sum_{(\beta:\nu)\le k}|\xi^\beta|,$$

where $\omega_6(\varepsilon) \to 0$ when $\varepsilon \to 0$.

By appropriately choosing ε, we obtain that with some constant $C_3 > 0$ it holds

$$C_3 \sum_{(\alpha:\nu)\le k+1} |\xi^\alpha| \le \Big|\sum_{(\alpha:\nu)\le 1} \tilde a^0_\alpha \xi^\alpha\Big| \sum_{(\beta:\nu)\le k} |\xi^\beta|.$$

From this inequality, using the [29, estimates (2.12)], we obtain that there exists a constant $\delta > 0$ such that

$$\Big|\sum_{(\alpha:\nu)\le 1} \tilde a^0_\alpha \xi^\alpha\Big| \ge \delta(1+|\xi|_\nu).$$

Since the last inequality holds for all the partial limits of sequences $\{a^0_\alpha(x_m) : (\alpha : \nu) \le 1\}$, where $|x_m| \to \infty$ when $m \to \infty$, we conclude the existence of constants $\delta > 0$ and $M > 0$ such that

$$\Big|\sum_{(\alpha:\nu)\le 1} a^0_\alpha(x) \xi^\alpha\Big| \ge \delta(1+|\xi|_\nu), \forall \xi \in \mathbb{R}^n, |x| > M.$$

□

It turns out that the necessary conditions obtained on the symbol of operators are also sufficient for fulfilling the a priori estimates (2.4) in the considered spaces.

Theorem 2.4. *Let $k \in \mathbb{Z}_+, q \in Q^{k,\mathcal{R}}$, and $P(x,\mathbb{D})$ be the differential form given by* (2.1) *with the coefficients satisfying* $\lim_{m\to\infty} \max_{x,y\in\overline{W_m}} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ *for all $\alpha \in \mathcal{R}$. Assume $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$, and there exist constants $\delta > 0$ and $M > 0$ such that*

$$\Big|\sum_{\alpha\in\mathcal{R}} a^0_\alpha(x) \xi^\alpha\Big| \ge \delta(1+|\xi|_{\partial\mathcal{R}}), \quad \forall \xi \in \mathbb{R}^n, |x| > M. \tag{2.20}$$

Then there exist constants $\kappa > 0$ and $N > 0$ such that

$$\|u\|_{k+1,\mathcal{R},p,q} \le \kappa(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_N)}), \forall u \in H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n). \tag{2.21}$$

Theorem 2.5. *Let $k \in \mathbb{Z}_+, q \in \widetilde Q^{k,\nu}$, and $P(x,\mathbb{D})$ be the differential form given by* (2.2) *with the coefficients satisfying* $\lim_{m\to\infty} \max_{x,y\in\overline{W_m}} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ *for all $\alpha \in \mathbb{Z}^n_+, (\alpha : \nu) \le 1$. Assume $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$, and there exist constants $\delta > 0$ and $M > 0$ such that*

$$\Big|\sum_{(\alpha:\nu)\le 1} a^0_\alpha(x) \xi^\alpha\Big| \ge \delta(1+|\xi|_\nu), \quad \forall \xi \in \mathbb{R}^n,\ |x| > M. \tag{2.22}$$

Then there exist constants $\kappa > 0$ and $N > 0$ such that

$$\|u\|_{k+1,\nu,p,q} \le \kappa(\|Pu\|_{k,\nu,p,q} + \|u\|_{L_p(K_N)}), \quad \forall u \in H^{k+1,\nu,p}_q(\mathbb{R}^n). \tag{2.23}$$

Proof. We combine the proof of these two theorems for completely regular polyhedron $\mathcal{R}$. When necessary, a distinction between the weight functions from $\widetilde Q^{k,\nu}$ and $Q^{k,\mathcal{R}}$ and the corresponding spaces $H^{k+1,\nu,p}_q(\mathbb{R}^n)$ and $H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n)$ are provided. Let $m_0 \in \mathbb{N}$. Using the a priori estimates for bounded domains from the work [23] with some constants $C_1 > 0$ and $N_1 > 0$ we have

$$\sum_{m=1}^{m_0} \|\varphi_m u\|_{k+1,\mathcal{R},p,q} \le C_1\Big(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_{N_1})}\Big), \tag{2.24}$$

for all $u \in H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n)$, where N_1 is such that $\bigcup_{i=1}^{m_0} W_i \subset K_{N_1}$. We denote

$$P^m(x,\mathbb{D}) := \sum_{\alpha\in\mathcal{R}} \Big[\psi_m(x)\Big(a^0_\alpha(x)q(x)^{1-\max_i(\alpha:\mu^i)} - a^0_\alpha(x_m)q(x_m)^{1-\max_i(\alpha:\mu^i)}\Big)$$

$$+ a^0_\alpha(x_m) q(x_m)^{1-\max_i(\alpha:\mu^i)}\Big] D^\alpha, \quad m = 1, 2, \dots.$$

Taking into consideration that $q \in Q^{k,\mathcal{R}}$ and the properties of $\{\varphi_m\}_{m=1}^\infty$, we obtain that for all $\gamma \in k\mathcal{R}$ and $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ and $m_0(\varepsilon) > 0$ such that for all $m > m_0$ and $\max_{j=1,\dots,l} \operatorname{diam} U_j < \delta$ the following inequality holds

$$\frac{|D^\gamma \varphi_m(x)|}{q(x)^{(\gamma:\mu_i)}} = \frac{|D^\gamma \varphi_m(x)| a^{|\gamma|}_{[\frac{m-1}{l}]}}{q(x)^{(\gamma:\mu_i) - \frac{|\gamma|}{\mu_{max}}} q(x)^{\frac{|\gamma|}{\mu_{max}}} a^{|\gamma|}_{[\frac{m-1}{l}]}} \le \tau_{1,\gamma}(\varepsilon), \tag{2.25}$$

where $\tau_{1,\gamma}(\varepsilon) \to 0$ when $\varepsilon \to 0$. For $q \in \widetilde{Q}^{k,\nu}$ and $(\gamma : \nu) \le k$, we have

$$\frac{|D^\gamma \varphi_m(x)|}{q(x)^{(\gamma:\nu)}} = \frac{|D^\gamma \varphi_m(x)| a^{|\gamma|}_{[\frac{m-1}{l}]}}{q(x)^{-(\frac{|\gamma|}{\nu_{\min}} - (\gamma:\nu))} q(x)^{\frac{|\gamma|}{\nu_{\min}}} a^{|\gamma|}_{[\frac{m-1}{l}]}} \le \tau_{2,\gamma}(\varepsilon), \tag{2.26}$$

where $\tau_{2,\gamma}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The corresponding inequalities apply to the functions $\{\psi_m\}_{m=1}^\infty$ in a similar manner. Using these estimates, the conditions on the coefficients

$$\lim_{m\to\infty} \max_{x,y \in \overline{W_m}}, |a^0_\alpha(x) - a^0_\alpha(y)| = 0$$

and [28, Lemma 3.1], analogously to the proof [17, Theorem 2.2], one can verify that for a sufficiently large m_0 and $m > m_0$, the operators $P^m(x, \mathbb{D}) : H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k,\mathcal{R},p}_q(\mathbb{R}^n)$ have bounded inverse operators. Since (2.5) holds, they have uniformly bounded norms, and with some $C_2 > 0$ it holds

$$\|\varphi_m u\|_{k+1,\mathcal{R},p,q} \le C_2 \|P^m(\varphi_m u)\|_{k,\mathcal{R},p,q}, \quad \forall u \in H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n).$$

Since $P^m(\varphi_m u) = P_0(\varphi_m u)$, for all $u \in H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n)$ and $m > m_0$, we obtain

$$\begin{aligned}\|\varphi_m u\|_{k+1,\mathcal{R},p,q} &\le C_2 \|P^m(\varphi_m u)\|_{k,\mathcal{R},p,q} \\ &\le C_2 \|P_0(\varphi_m u)\|_{k,\mathcal{R},p,q}, \quad \forall u \in H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n).\end{aligned}$$

Using the properties of the functions $\{\varphi_m\}_{m=1}^\infty$ and estimate (2.25) for $q \in Q^{k,\mathcal{R}}$ and (2.26) for $q \in \widetilde{Q}^{k,\nu}$, it can be shown that for a sufficiently large m_0 and a small enough $\max_{j=1,\dots,l} \operatorname{diam} U_j$ for $m > m_0$ with some constants $C_3, C_4 > 0$ the following estimate holds:

$$\begin{aligned}&\|\varphi_m P_0 u - P_0(\varphi_m u)\|^p_{k,\mathcal{R},p,q} \\ &\le C_3 \Big\| \sum_{\alpha\in\mathcal{R}} \sum_{\beta+\gamma=\alpha, |\gamma|>0} a^0_\alpha(x) D^\beta u D^\gamma \varphi_m q(x)^{1-\max_i(\alpha:\mu^i)} \Big\|^p_{k,\mathcal{R},p,q} \\ &\le C_4 \Big\| \sum_{\alpha\in\mathcal{R}} \sum_{\beta+\gamma=\alpha, |\gamma|>0} a^0_\alpha(x) D^\beta u D^\gamma \varphi_m \frac{1}{q(x)^{\min_i(\gamma:\mu^i)}} q(x)^{1-\max_i(\beta:\mu^i)} \Big\|^p_{k,\mathcal{R},p,q} \\ &\le \omega_1(\varepsilon) \|u\|^p_{H^{k+1,\mathcal{R},p}_q(W_m)},\end{aligned}$$

where $\omega_1(\varepsilon) \to 0$ when $\varepsilon \to 0$.

Summing up for all $m > m_0$ and taking into account the property (ii) of $\{\varphi_m\}_{m=1}^\infty$ and $\{W_m\}_{m=1}^\infty$, with some constant $C_5 > 0$ we obtain

$$\sum_{m=m_0+1}^\infty \|\varphi_m u\|^p_{k+1,\mathcal{R},p,q} \le C_5(\|P_0 u\|^p_{k,\mathcal{R},p,q} + \omega_1(\varepsilon) \|u\|^p_{k+1,\mathcal{R},p,q}), \tag{2.27}$$

for all $u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n)$. Using the properties of the functions $\{\varphi_m\}_{m=1}^{\infty}$, along with (2.24) and (2.27), we establish that with some constant $C_6 > 0$ the following holds:

$$\begin{aligned}\|u\|_{k+1,\mathcal{R},p,q} &\le \sum_{m=1}^{m_0} \|\varphi_m u\|_{k+1,\mathcal{R},p,q} + \sum_{m=m_0+1}^{\infty} \|\varphi_m u\|_{k+1,\mathcal{R},p,q} \\ &\le C_6\Big(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_{N_1})} + \|P_0 u\|_{k,\mathcal{R},p,q} \\ &\quad + \omega_2(\varepsilon)\|u\|_{k+1,\mathcal{R},p,q}\Big), \quad \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n),\end{aligned} \tag{2.28}$$

where $\omega_2(\varepsilon) \to 0$ as $\varepsilon \to 0$.

We have $P_0(x,\mathbb{D}) = P(x,\mathbb{D}) - L(x,\mathbb{D})$. Then

$$\|P_0 u\|_{k,\mathcal{R},p,q} \le \|Pu\|_{k,\mathcal{R},p,q} + \|Lu\|_{k,\mathcal{R},p,q}, \quad \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n).$$

Considering the conditions $D^\beta(a_\alpha^1(x)) = o(q(x)^{1-\max_i(\alpha-\beta:\mu^i)})$ when $|x| \to \infty$, $\alpha \in \mathcal{R}, \beta \in k\mathcal{R}$ it can be verified that for a sufficiently large m_0

$$\|Lu\|_{k,\mathcal{R},p,q} \le \omega_3(\varepsilon)\|u\|_{k+1,\mathcal{R},p,q} + C_7\|u\|_{H^{k+1,\mathcal{R},p}(K_{N_1})}, \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n),$$

where $\omega_3(\varepsilon) \to 0$ when $\varepsilon \to 0$ and N_1 is such that $\bigcup_{i=1}^{m_0} W_i \subset K_{N_1}$.

Similarly to (2.24), applying the a priori estimate from [23], with some constant $C_8 > 0$, we obtain

$$\|Lu\|_{k,\mathcal{R},p,q} \le \omega_3(\varepsilon)\|u\|_{k+1,\mathcal{R},p,q} + C_8(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_{N_1})}).$$

Combining the last estimate with (2.28), we obtain

$$\begin{aligned}&\|u\|_{k+1,\mathcal{R},p,q} \\ &\le C_6(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_{N_1})} + \|P_0 u\|_{k,\mathcal{R},p,q} + \omega_2(\varepsilon)\|u\|_{k+1,\mathcal{R},p,q}) \\ &\le C_6\Big(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_{N_1})} + \omega_3(\varepsilon)\|u\|_{k+1,\mathcal{R},p,q} \\ &\quad + C_8(2\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_{N_1})}) + \omega_2(\varepsilon)\|u\|_{k+1,\mathcal{R},p,q}\Big), \quad \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n).\end{aligned}$$

Choosing m_0 sufficiently large and $\max_{j=1,\dots,l} \operatorname{diam} U_j$ sufficiently small such that

$$C_6(\omega_3(m_0) + \omega_2(m_0)) < 1/2,$$

then, with some constant $C_9 > 0$, we obtain

$$\|u\|_{k+1,\mathcal{R},p,q} \le C_9\Big(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_{N_1})}\Big), \quad \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n).$$

□

Corollary 2.6. *Let $k \in \mathbb{N}, q \in \widetilde{Q}^{k,\nu}$, and $P(x,\mathbb{D})$ be the differential form given by (2.2) with the coefficients satisfying $\lim_{m\to\infty} \max_{x,y\in\overline{W_m}} |a_\alpha^0(x) - a_\alpha^0(y)| = 0$ for all $\alpha \in \mathbb{Z}_+^n, (\alpha:\nu) \le 1$. Assume $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$, and there exist constants $\delta > 0$ and $M > 0$ such that*

$$\Big|\sum_{(\alpha:\nu)\le 1} a_\alpha^0(x)\xi^\alpha\Big| \ge \delta(1 + |\xi|_\nu), \forall \xi \in \mathbb{R}^n, |x| > M. \tag{2.29}$$

Then, if $u \in H_q^{k,\nu,p}(\mathbb{R}^n), P(x,\mathbb{D})u \in H_q^{k,\nu,p}(\mathbb{R}^n)$, then $u \in H_q^{k+1,\nu,p}(\mathbb{R}^n)$.

Proof. Applying Theorem 2.5, we obtain that with some constant $C_1 > 0$ for $u \in H_q^{k,\nu,p}(\mathbb{R}^n)$ the following estimate holds

$$\|u\|_{k,\nu,p,q} \le C_1(\|Pu\|_{k-1,\nu,p,q} + \|u\|_{L_p(K_N)}). \tag{2.30}$$

Using arguments similar to those in [1, Theorem 15.1], one can check that there exists a constant $C_2 > 0$ such that

$$\sum_{(\alpha:\nu)=k+1} \|D^\alpha u\|_{L_p(\mathbb{R}^n)} \le C_2\Big(\|Pu\|_{k,\nu,p,q} + \|u\|_{H^{1,\nu,p}(K_N)}\Big). \tag{2.31}$$

Using the property 1 of $q \in \widetilde{Q}^{k,\nu}$ and estimate (2.30) with some constant $C_3 > 0$, we obtain

$$\begin{aligned} \sum_{(\alpha:\nu)\le k} \|D^\alpha u \cdot q^{k+1-(\alpha:\nu)}\|_{L_p(\mathbb{R}^n)} &\le C_3 \sum_{(\alpha:\nu)\le k} \|D^\alpha u \cdot q^{k-(\alpha:\nu)}\|_{L_p(\mathbb{R}^n)} \\ &\le C_4(\|Pu\|_{k-1,\nu,p,q} + \|u\|_{L_p(K_N)}). \end{aligned} \tag{2.32}$$

Taking into consideration that $u \in H_q^{k,\nu,p}(\mathbb{R}^n)$ and the fact that $H_q^{k,\nu,p}(\mathbb{R}^n)$ is embedded in $H^{1,\nu,p}(K_N)$, from (2.31) and (2.32), we obtain that $u \in H_q^{k+1,\nu,p}(\mathbb{R}^n)$. $\square$

3. Fredholm criteria

Lemma 3.1. *For $k \in \mathbb{Z}_+$ and $q \in Q^{k,\mathcal{R}}$, the space $H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n)$ is compactly embedded in $H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$.*

Proof. Since $\frac{1}{q(x)} \rightrightarrows 0$ as $|x| \to \infty$, for each $\varepsilon > 0$ there exist $N = N(\varepsilon) > 0$ and $\phi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\phi_\varepsilon \subset K_N$, $0 \le \phi_\varepsilon(x) \le 1$ for all $x \in \mathbb{R}^n$, $\phi_\varepsilon(x) = 1$ for $|x| \le N/2$ and $\phi_\varepsilon(x) = 0$ for $|x| \ge N$ that with some constants $C_1 = C_1(\varepsilon) > 0, C_2 = C_2(\varepsilon) > 0$ the following estimate holds

$$\begin{aligned} &\|u\|_{k,\mathcal{R},p,q} \\ &= \sum_{\alpha\in k\mathcal{R}} \|D^\alpha u \cdot q^{k-\max_i(\alpha:\mu^i)}\|_{L_p(\mathbb{R}^n)} \\ &= \sum_{\alpha\in k\mathcal{R}} \|D^\alpha u((1-\phi_\varepsilon(x))\frac{1}{q(x)}q(x)^{k+1-\max_i(\alpha:\mu^i)} + \phi_\varepsilon(x)q(x)^{k-\max_i(\alpha:\mu^i)})\|_{L_p(\mathbb{R}^n)} \\ &\le \varepsilon \sum_{\alpha\in(k+1)\mathcal{R}} \|(1-\phi_\varepsilon)D^\alpha u \cdot q^{k+1-\max_i(\alpha:\mu^i)}\|_{L_p(\mathbb{R}^n)} + C_1\|\phi_\varepsilon u\|_{H_q^{k,\mathcal{R},p}(K_N)} \\ &\le \varepsilon\|u\|_{k+1,\mathcal{R},p,q} + C_2\|\phi_\varepsilon u\|_{\dot{H}^{k,\mathcal{R},p}(K_N)}. \end{aligned}$$

Since $\dot{H}^{k+1,\mathcal{R},p}(K_N)$ is compactly embedded in $\dot{H}^{k,\mathcal{R},p}(K_N)$, applying the previous estimate and [2, Proposition 10.8], we obtain that there exists a constant $C_3 = C_3(\varepsilon) > 0$ such that

$$\|u\|_{k,\mathcal{R},p,q} \le \tau(\varepsilon)\|u\|_{k+1,\mathcal{R},p,q} + C_3\|u\|_{L_p(K_N)}, \quad \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n), \tag{3.1}$$

where $\tau(\varepsilon) \to 0$ as $\varepsilon \to 0$.

From (3.1) and [2, Proposition 10.8], we obtain that $H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n)$ is compactly embedded in $H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$. $\square$

Further, we use the following theorem (see [12, Theorem 3.14]).

Theorem 3.2. *Let A be a bounded linear operator acting from a Banach space X to a Banach space Y. Then the following holds*

(1) *if the operator A has a left regularizer, then kernel of operator A in X is finite dimensional;*
(2) *if the operator A has a right regularizer, then the image of operator A is closed in Y and the cokernel is finite dimensional;*
(3) *the operator A has left and right regularizers if and only if A is a Fredholm operator.*

Theorem 3.3. *Let $k \in \mathbb{Z}_+, q \in Q^{k,\mathcal{R}}$, and $P(x,\mathbb{D})$ be the differential operator given by (2.1) with the coefficients satisfying $\lim_{m\to\infty} \max_{x,y\in\overline{W_m}} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for all $\alpha \in \mathcal{R}$. Then, the operator $P(x,\mathbb{D}) : H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k,\mathcal{R},p}_q(\mathbb{R}^n)$ is a Fredholm operator if and only if $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that*

$$|\sum_{\alpha\in\mathcal{R}} a^0_\alpha(x)\xi^\alpha| \geq \delta(1 + |\xi|_{\partial\mathcal{R}}), \quad \forall \xi \in \mathbb{R}^n,\ |x| > M. \tag{3.2}$$

Theorem 3.4. *Let $k \in \mathbb{Z}_+, q \in \widetilde{Q}^{k,\nu}$, and $P(x,\mathbb{D})$ be the differential operator given by (2.2) with the coefficients satisfying $\lim_{m\to\infty} \max_{x,y\in\overline{W_m}} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for all $\alpha \in \mathbb{Z}^n_+, (\alpha : \nu) \leq 1$. Then, the operator $P(x,\mathbb{D}) : H^{k+1,\nu,p}_q(\mathbb{R}^n) \to H^{k,\nu,p}_q(\mathbb{R}^n)$ is a Fredholm operator if and only if $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that*

$$\Big|\sum_{(\alpha:\nu)\leq 1} a^0_\alpha(x)\xi^\alpha\Big| \geq \delta(1 + |\xi|_\nu), \quad \forall \xi \in \mathbb{R}^n,\ |x| > M. \tag{3.3}$$

Proof. Since a Fredholm operator is n-normal, the necessary part is a consequence of Theorem 2.1 and Theorem 2.2 for Theorem 3.3, and Theorem 2.3 for Theorem 3.4.

Now, let us prove the sufficient part. We combine the proofs of these two theorems for completely regular polyhedron $\mathcal{R}$. When necessary, a distinction between the weight functions from $\widetilde{Q}^{k,\nu}$ and $Q^{k,\mathcal{R}}$ and the corresponding spaces $H^{k+1,\nu,p}_q(\mathbb{R}^n)$ and $H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n)$ will be provided. Applying Theorem 2.4, we conclude that the operator $P(x,\mathbb{D}) : H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k,\mathcal{R},p}_q(\mathbb{R}^n)$ is n-normal. It remains to prove that the cokernel of the operator $P(x,\mathbb{D}) : H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k,\mathcal{R},p}_q(\mathbb{R}^n)$ is finite dimensional.

Let $m_0 \in \mathbb{N}$ and $x_m \in W_m, m = 1, 2 \dots$. For $m \leq m_0$, we denote

$$P^m(x,\mathbb{D}) := \sum_{\alpha\in\mathcal{R}} (\psi_m(x)(a_\alpha(x) - a_\alpha(x_m)) + a_\alpha(x_m))D^\alpha,$$
$$P^{m,0}(x,\mathbb{D}) := \sum_{\alpha\in\partial\mathcal{R}} (\psi_m(x)(a_\alpha(x) - a_\alpha(x_m)) + a_\alpha(x_m))D^\alpha,$$
$$R^{m,0} := F^{-1} \frac{|\xi|_{\partial\mathcal{R}}}{(1 + |\xi|_{\partial\mathcal{R}})P^{m,0}(x_m,\xi)} F.$$

Since $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$, for sufficiently small diameters of $\{W_m\}^{m_0}_{m=1}$, from [27, Lemma 3.1], it follows that for $m \leq m_0$ the following representation holds:

$$P^m(x,\mathbb{D})R^{m,0} = I + T^m_1 + T^m_2, \tag{3.4}$$

where $T_1^m : H^{k,\mathcal{R},p}(\mathbb{R}^n) \to H^{k+\sigma,\mathcal{R},p}(\mathbb{R}^n)$ with $\sigma = \sigma(\mathcal{R}) > 0$ and the operator $T_2^m : H^{k,\mathcal{R},p}(\mathbb{R}^n) \to H^{k,\mathcal{R},p}(\mathbb{R}^n)$ satisfies $\|T_2^m\| < 1$. We denote

$$R^m := R^{m,0}(I + T_2^m)^{-1}.$$

Then

$$P^m R^m = I + T^m, \tag{3.5}$$

where $T^m : H^{k,\mathcal{R}}(\mathbb{R}^n) \to H^{k+\sigma,\mathcal{R}}(\mathbb{R}^n)$ with some $\sigma = \sigma(\mathcal{R}) > 0$.

For $m > m_0$, we denote

$$\begin{aligned} P^m(x,\mathbb{D}) := \sum_{\alpha\in\mathcal{R}} \Big[&\psi_m(x)\Big(a_\alpha^0(x)q(x)^{1-\max_i(\alpha:\mu^i)} - a_\alpha^0(x_m)q(x_m)^{1-\max_i(\alpha:\mu^i)}\Big) \\ &+ a_\alpha^0(x_m)q(x_m)^{1-\max_i(\alpha:\mu^i)}\Big] D^\alpha. \end{aligned}$$

Similarly as in the proof of Theorem 2.4, we can take m_0 large enough such that for $m > m_0$ operators $P^m : H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ have uniformly bounded inverse operators $R^m : H_q^{k,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n)$. Consider

$$Rf := \sum_{l=0}^{\infty} \psi_l R^l(\varphi_l f), f \in H_q^{k,\mathcal{R},p}(\mathbb{R}^n). \tag{3.6}$$

Similarly to the proof of [27, Theorem 2.6], it can be checked that the following representation holds

$$P(x,\mathbb{D})Ru = u + \phi T_1 u + T_2 u,$$

where $\phi \in C_0^\infty(\mathbb{R}^n)$, $T_1 : H^{k,\mathcal{R},p}(\mathbb{R}^n) \to H^{k+\sigma,\mathcal{R},p}(\mathbb{R}^n)$ with $\sigma = \sigma(\mathcal{R}) > 0$ and operator $T_2 : H_q^{k,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ satisfies $\|T_2\| < 1$.

For $q \in Q^{k,\mathcal{R}}$, by using Lemma 3.1 and [27, Theorem 2.6], it can be checked that $R : H_q^{k,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n)$ is a right regularizer. Let us prove this for the case $q \in \widetilde{Q}^{k,\nu}$. Since $\phi \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}\phi \subset K_{N_1}$ with some $N_1 > 0$ and $T_1 : H^{k,\nu,p}(\mathbb{R}^n) \to H^{k+\sigma,\nu,p}(\mathbb{R}^n)$ with $\sigma = \sigma(\nu) > 0$, there exist constants $C_1, C_2, C_3, C_4 > 0$ such that

$$\begin{aligned} \|\phi T_1 u\|_{k,\nu,p,q} &\le C_1 \|\phi T_1 u\|_{\dot H^{k,\nu,p}(K_{N_1})} \le C_2 \|\phi T_1 u\|_{\dot H^{k+\sigma,\nu,p}(K_{N_1})} \\ &\le C_3 \|u\|_{\dot H^{k,\nu,p}(K_{N_1})} \le C_4 \|u\|_{k,\nu,p,q}, \forall u \in H_q^{k,\nu,p}(\mathbb{R}^n). \end{aligned}$$

Consider the bounded sequence $\{u_n\}_{n=1}^\infty \subset H_q^{k,\nu,p}(\mathbb{R}^n)$. Using the previous estimate and the fact that $\dot H^{k+\sigma,\nu,p}(K_{N_1})$ is compactly embedded in $\dot H^{k,\nu,p}(K_{N_1})$, we have a convergent subsequence of $\{\phi T_1 u_n\}_{n=1}^\infty$. Thus, the operator $\phi T_1 : H_q^{k,\nu,p}(\mathbb{R}^n) \to H_q^{k,\nu,p}(\mathbb{R}^n)$ is compact.

Since the operator $T_2 : H_q^{k,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ satisfies $\|T_2\| < 1$, there exists $(I + T_2)^{-1}$. Applying this operator to both sides, we obtain

$$P(x,\mathbb{D})\widetilde{R}u = u + \widetilde{T}u,$$

where $\widetilde{R} := R(I + T_2)^{-1}$ and $\widetilde{T} := \phi T_1(I + T_2)^{-1} : H_q^{k,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ is a compact operator since ϕT_1 is a compact. Then, applying Theorem 3.2, we conclude that the cokernel of the operator $P(x,\mathbb{D}) : H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ is finite dimensional. Therefore, the operator $P(x,\mathbb{D}) : H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ is a Fredholm operator. $\square$

4. Properties on the scales of weighted spaces

We denote

$$\begin{gathered}
\ker(P; H_q^{k,\mathcal{R},p}) := \{u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n) : P(x,\mathbb{D})u = 0\}, \\
\operatorname{Im}(P; H_q^{k,\mathcal{R},p}) := \{f \in H_q^{k,\mathcal{R},p}(\mathbb{R}^n) : \exists u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n) \text{ s.t. } P(x,\mathbb{D})u = f\}, \\
\operatorname{coker}(P; H_q^{k,\mathcal{R},p}) := H_q^{k,\mathcal{R},p}(\mathbb{R}^n)/\overline{\operatorname{Im}(P; H_q^{k,\mathcal{R},p})}, \\
\operatorname{ind}(P; H_q^{k,\mathcal{R},p}) := \dim\ker(P; H_q^{k,\mathcal{R},p}) - \dim\operatorname{coker}(P; H_q^{k,\mathcal{R},p}).
\end{gathered}$$

Corollary 4.1. *Let $k \in \mathbb{Z}_+, q \in Q^{k,\mathcal{R}}$, and $P(x,\mathbb{D})$ be the differential operator* (2.1)*. Assume* (3.2) *holds and the coefficients satisfy* $\lim_{m\to\infty}\max_{x,y\in\overline{W_m}}|a_\alpha^0(x) - a_\alpha^0(y)| = 0$ *for all* $\alpha \in \mathcal{R}$*. Then* $\ker(P; H_q^{k,\mathcal{R},p})$, $\operatorname{coker}(P; H_q^{k,\mathcal{R},p})$*, and* $\operatorname{ind}(P; H_q^{k,\mathcal{R},p})$ *are independent of k and p.*

Proof. The analogous construction (3.6) can be done for the left regularizer. Since the left regularizer exists, using [28, Corollary 3.2], we obtain that for $k_1, k_2 \in \mathbb{Z}_+$ the following equalities hold: $\ker(P; H_q^{k_1,\mathcal{R},p}) = \ker(P; H_q^{k_2,\mathcal{R},p})$, $\operatorname{coker}(P; H_q^{k_1,\mathcal{R},p}) = \operatorname{coker}(P; H_q^{k_2,\mathcal{R},p})$, $\operatorname{ind}(P; H_q^{k_1,\mathcal{R},p}) = \operatorname{ind}(P; H_q^{k_2,\mathcal{R},p})$. So, we establish the independence from k. So, for $k \in \mathbb{Z}_+$ we have $Ker(P; H_q^{k,\mathcal{R},p}) \subset \bigcap_{s\geq 0} H_q^{s,\mathcal{R},p}(\mathbb{R}^n)$. Since $\frac{1}{q(x)} \rightrightarrows 0$ when $|x| \to \infty$, it is easy to show that $\bigcap_{s\geq 0} H_q^{s,\mathcal{R},p}(\mathbb{R}^n) \subset S$. Thus, the kernel is independent of k and p. Analogously, this is true for the kernel of adjoint operator. Then, using [18, Theorem 3.1], we obtain the independence from k and p for the cokernel. Therefore, the index of the operator is also independent of k and p. □

Corollary 4.2. *Let $P(x,\mathbb{D}) : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ be an operator from Theorem 3.3, considered as an unbounded operator in $L_2(\mathbb{R}^n)$, and assume that* (3.2) *holds. Then, one of the following statements holds:*

- $\sigma(P) = \mathbb{C}$*;*
- $\sigma(P)$ *is discrete and* $ind(P; H_q^{k,\mathcal{R}}) = 0$.

Proof. From Lemma 3.1 and Theorem 3.3, it follows that, for every $\lambda \in \mathbb{C}$, the operator $P(x,\mathbb{D}) - \lambda I : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ is a Fredholm operator. Then, utilizing arguments similar to those in [25, Theorem 8.4] and taking into account Lemma 3.1, we establish that one of the statements from the corollary is true. □

For the case $q \in \widetilde{Q}^{k,\nu}$, the properties on the scale of spaces $H_q^{k,\nu,p}(\mathbb{R}^n)$ can differ from the previous class. Further, we consider the case $q \equiv 1$ and $p = 2$.

Corollary 4.3. *Let $q \equiv 1$ and $P(x,\mathbb{D})$ be the differential operator* (2.2)*. Assume that* (3.3) *holds and the coefficients satisfy* $\lim_{m\to\infty}\max_{x,y\in\overline{W_m}}|a_\alpha^0(x) - a_\alpha^0(y)| = 0$ *for all* $(\alpha : \nu) \leq 1$*. Then* $\ker(P; H^{k,\nu})$, $\operatorname{coker}(P; H^{k,\nu})$*, and* $\operatorname{ind}(P; H^{k,\nu})$ *are independent of k.*

Proof. Using Corollary 2.6, it is easy to check that for $k_1, k_2 \in \mathbb{Z}_+$ the following equality holds: $\ker(P; H^{k_1,\nu}) = \ker(P; H^{k_2,\nu})$. Then, from Corollary 2.6 and [30, Lemma 2.1], we obtain a similar equality for the cokernels. Therefore, the kernels, cokernels, and, consequently, the index of the operator are independent of k. □

Corollary 4.4. *Let $q \equiv 1$ and $P(x,\mathbb{D}) : H^{k+1,\nu}(\mathbb{R}^n) \to H^{k,\nu}(\mathbb{R}^n)$ be an operator from Theorem 3.4, considered as an unbounded operator in $L_2(\mathbb{R}^n)$. Let there exist constants $\widetilde{a}_\alpha$ such that $a_\alpha(x) \rightrightarrows \widetilde{a}_\alpha$ when $|x| \to \infty$, $(\alpha : \nu) \le 1$. Then*

$$\sigma_{es}(P) = \Big\{ \sum_{(\alpha:\nu)\le 1} \widetilde{a}_\alpha \xi^\alpha : \xi \in \mathbb{R}^n \Big\}.$$

Proof. From Theorem 3.4 and conditions on the coefficients, we obtain that operator $P(x,\mathbb{D}) - \lambda I : H^{k+1,\nu}(\mathbb{R}^n) \to H^{k,\nu}(\mathbb{R}^n)$ is a Fredholm operator if and only if there exists a constant $\delta > 0$ such that

$$\Big| \sum_{(\alpha:\nu)\le 1} \widetilde{a}_\alpha \xi^\alpha - \lambda \Big| \ge \delta(1 + |\xi|_\nu). \tag{4.1}$$

It follows from (4.1) that $\sigma_{es}(P) = \big\{ \sum_{(\alpha:\nu)\le 1} \widetilde{a}_\alpha \xi^\alpha : \xi \in \mathbb{R}^n \big\}$. ☐

Remark 4.5. Using condition (4.1) and [25, Proposition 8.1], it is easy to verify that, for $\lambda \notin \sigma_{es}(P)$, the index of the operator $P(x,\mathbb{D}) - \lambda I : H^{k+1,\nu}(\mathbb{R}^n) \to H^{k,\nu}(\mathbb{R}^n)$ from Corollary 4.4 is 0, but the dimensions of the kernel and the cokernel for such operators may differ from 0 (see examples from [9]).

Acknowledgements. This work was partially supported by the Thematic Funding of the Russian - Armenian University and the Ministry of Education and Science of the Russian Federation, and by the Science Committee of the Republic of Armenia, scientific project N25RG-1A205.

References

[1] S. Agmon, A. Douglis, L. Nirenberg; *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*. I. Communications on Pure and Applied Mathematics, 12 (1959), 623–727.

[2] M. S. Agranovich; *Elliptic singular integro-differential operators*. Russian Math. Surveys, 20:5 (1965), 1–121.

[3] L. A. Bagirov; *Elliptic operators in unbounded domain*. Sbornik: Mathematics, 86:1 (1971), 122–139 (in Russian).

[4] L. A. Bagirov; *A priori estimates, existence theorems, and the behavior at infinity of solutions of quasielliptic equations in $\mathbb{R}^n$*. Sbornik: Mathematics. 110:4 (1979), 475–492.

[5] P. Boggiatto, E. Buzano E., L. Rodino; *Multi-quasi-elliptic operators in $\mathbb{R}^n$*. Partial Differential Operators and Mathematical Physics, Proceedings Holzau, (1995), 31–42.

[6] E. Buzano, A. Ziggioto; *Weyl formula for multi-quasi-elliptic operators. of Schrödinger type*. Annali di Matematica, 180 (2001), 223–243.

[7] E. Buzano, A. Ziggioto; *On the essential spectrum of hypoelliptic pseudodifferential operators*. Mathematische Nachrichten, 281:1 (2008), 5–24.

[8] A. A. Darbinyan, A. G. Tumanyan; *On a priori estimates and the Fredholm property of differential operators in anisotropic spaces*. Journal of Contemporary Mathematical Analysis, 53:2 (2018), 61–70.

[9] A. A. Darbinyan, A. G. Tumanyan; *On index stability of Noetherian differential operators in anisotropic Sobolev spaces*. Eurasian Mathematical Journal, 10:1 (2019), 9–15.

[10] G. V. Demidenko; *On quasielliptic operators in $\mathbb{R}^n$*. Siberian Mathematical Journal, 39:5 (1998), 1028–1037.

[11] G. V. Demidenko; *Quasielliptic operators and equations not solvable with respect to the highest order derivative*. J. Math. Sci., 230:1 (2018), 25–35.

[12] D. E. Edmunds, W. D. Evans; *Spectral theory and differential operators*. Oxford University Press, Oxford, 1987.

[13] J. Friberg; *Multi-quasi-elliptic polynomials*. Ann. Scuola Norm. Sup. Pisa C1. Sc., 21 (1967), 239–260.

[14] H. G. Ghazaryan; *The Newton polyhedron, spaces of differentiable functions and general theory of differential equations.* Armenian Journal of Mathematics, 9:2 (2017), 102–145.
[15] G. N. Hile; *Fundamental solutions and mapping properties of semielliptic operators.* Math. Nachrichten, 279:13–14 (2006), 1538–1572.
[16] L. Hörmander; *Linear Partial Differential Operators.* Springer-Verlag Berlin Heidelberg, 1969.
[17] G. A. Karapetyan, A. A. Darbinyan; *Index of semielliptic operator in* $\mathbb{R}^n$. Proceedings of the NAS Armenia:Mathematics, 42:5 (2007), 33–50.
[18] S. G. Kreyn; *Linear operators in Banach spaces.* Nauka, Moscow, 1971 (in Russian).
[19] R. B. Lockhart, R. C. McOwen; *On elliptic systems in* $\mathbb{R}^n$. Acta Math., 150 (1983), 125–135.
[20] R. C. McOwen; *On elliptic operators in* $\mathbb{R}^n$. Communications in Partial Differential Equations, 5:8 (1980), 913–933.
[21] V. P. Mikhailov; *Behavior at infinity of a certain class of polynomials.* Proc. Steklov Inst. Math., 91 (1967), 61–82.
[22] S. M. Nikolsky; *The first boundary-value problem for a general linear equation.* Dokl. Akad. Nauk SSSR, 146:4 (1962), 767–769 (in Russian).
[23] E. Pehkonen; *Ein hypoelliptisches Dirichlet Problem.* Com. Mat. Phys., 48:3 (1978), 131–143.
[24] E. Schrohe; *Spectral invariance, ellipticity, and the Fredholm property for pseudodifferential operators on weighted Sobolev spaces.* Annals of Global Analysis and Geometry, 10:3 (1992), 237–254.
[25] M. A. Shubin; *Pseudodifferential Operators and Spectral Theory.* Springer-Verlag, Berlin, 1987.
[26] A. G. Tumanyan; *A Priori Estimates and Fredholm Criteria for a Class of Regular Hypoelliptic Operators.* Siberian Advances in Mathematics, 33:2 (2023), 151–164.
[27] A. G. Tumanyan; *Fredholm criteria for a class of regular hypoelliptic operators in multianisotropic spaces in* $\mathbb{R}^n$. Italian Journal of Pure and Applied Mathematics, 48 (2022), 1009–1028.
[28] A. G. Tumanyan; *Fredholm property of regular hypoelliptic operators on the scales of multianisotropic spaces.* ITM Web of Conferences. International Conference on Applied Mathematics and Numerical Methods (ICAMNM 2022), 49 (2022), 1–13.
[29] A. G. Tumanyan; *On the Fredholm property of semielliptic operators in anisotropic weighted spaces in* $\mathbb{R}^n$. Journal of Contemporary Mathematical Analysis, 56:3 (2021), 168–181.
[30] A. G. Tumanyan; *On the invariance of index of semielliptical operator on the scale of anisotropic spaces.* Journal of Contemporary Mathematical Analysis, 51:4 (2016), 187–198.
[31] L. R. Volevich, S. G. Gindikin; *The Method of Newton's Polyhedron in the Theory of Partial Differential Equations.* Math. Appl, 86. Dordrecht, Kluwer Academic, 1992.

Ani Tumanyan
Institute of Mathematics, Russian-Armenian University, Armenia
Email address: ani.tumanyan@rau.am