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ON SOLUTIONS ARISING FROM RADIAL SPATIAL DYNAMICS OF SOME SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. We consider the semilinear elliptic equation

$$\Delta u + f(x,u) = 0,$$

where $x \in \mathbb{R}^N \setminus \{0\}$, $N \geq 2$, and f satisfies certain smoothness and structural assumptions. We construct solutions of the form $u(r,\phi) = r^{(2-N)/2}\tilde{u}(\log r, \phi)$, where $r = |x| > 0$, $\phi \in \mathbb{S}^{N-1}$, and $\tilde{u}$ is quasiperiodic in its first argument with two nonresonant frequencies. These solutions are found using some recent developments in the theory of spatial dynamics, in which the radial variable r takes the role of time, combined with classical results from dynamical systems and the KAM theory.

1. Introduction

We consider the semilinear elliptic equation

$$\Delta u + f_1(x,u) = 0, \quad x \in \mathbb{R}^N \setminus \{0\}, \tag{1.1}$$

where N is a positive integer, Δ is the Laplace operator in x, and $f_1 : (\mathbb{R}^N \setminus \{0\}) \times \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth function satisfying $f_1(\cdot, 0) \equiv 0$.

The study of geometrical properties of solutions of semilinear elliptic equations on the entire space $\mathbb{R}^N$ has been extensive. For instance, if one considers solutions decaying in all variables (also known as fully localized solutions), together with some assumptions on the nonlinearity, the classical result of Gidas, Ni, and Nirenberg [20] yields that all fully localized solutions are radially symmetric around some point in $\mathbb{R}^N$. On the other extreme, if no decay conditions are imposed, then a variety of solutions have been found, especially in the case of homogeneous problems (i.e., $f_1 = f_1(u)$). Just to give some examples we point to multi-bump solutions decaying along all but finitely many rays [30], saddle shaped solutions and general multiple-end solutions [17, 18, 28], as well as solutions having both fronts (transitions) and bumps [44].

Equation (1.1) is defined on a punctured domain. Such equations, along with equations on exterior domains, have been extensively studied as well. We mention only a few problems in this field: non-radial singular solutions to the Lane-Emden

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equation [14, 15], finite energy solutions in an exterior domain [6], equations involving supercritical exponents [13, 16], a priori estimates for solutions of superlinear elliptic equations and systems [37], a problem involving a singular nonlinearity [23], and the study of anisotropic singularities for a power nonlinearity [10]. Among the (very incomplete) list of references provided, [10, 14, 23] are of special relevance to us: their constructions are based on solving elliptic equations on spheres which are then used to obtain solutions on the punctured space. Our approach to construct solutions of (1.1) will be to some extent similar.

Among the wide variety of solutions of semilinear elliptic equations, one finds *quasiperiodic* solutions, which will be the focus of our attention in this paper. In previous articles [38, 40, 41], Poláčik and the author have studied the existence of solutions to some semilinear elliptic equations on the entire space with the following property: writing $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, the solutions constructed decay to 0 as $|x'| \to \infty$ uniformly in x_N, and are quasiperiodic (and not periodic) in x_N. Such solutions were found using a spatial dynamics approach to elliptic equations and results from the Kolmogorov-Arnold-Moser (KAM) theory [2, 27, 34]. Previously, related ideas for finding quasiperiodic solutions of elliptic equations on an unbounded strip have been used by Scheurle [46] and Valls [50] (see [38] for a more detailed discussion and further related references).

The main contribution of [38] is the outlining of a general scheme to find quasiperiodic solutions which, in principle, could be applied in other settings, yielding different conditions that may imply the existence of the desired quasiperiodic solutions. For instance, in [41] a different type of KAM theorem permitted the application of the general strategy from [38] to construct quasiperiodic solutions in such a way that the cubic terms (in u) of the nonlinear part of the equation are not involved in the usual nondegeneracy conditions: the nonlinearity may even be purely quadratic in some cases. (For another perspective on this issue and a KAM-type result for the Boussinesq equation with a quadratic nonlinearity see [48].) In [40] it is shown that the scheme can be applied to some homogeneous semilinear equations.

A common approach to spatial dynamics found in the literature applies to cylindrical domains of the form $\Omega \times \mathbb{R}$, with Ω a domain in $\mathbb{R}^{N-1}$ which is often, but not always, assumed to be bounded. The unbounded variable x_N takes the role of time, in the sense that the partial differential equation being considered is rewritten as an abstract equation in terms of x_N. In certain settings, such as elliptic problems, the Cauchy problem for the abstract equation is ill posed, yet in many situations it is still possible to find solutions. A number of authors have made contributions to the spatial dynamics approach to study partial differential equations, for instance, [9, 19, 22, 24, 26, 31, 32, 33, 35, 36, 51]. Several of the aforementioned works develop and make use of center manifold theory to successfully employ spatial dynamics, but other approaches can be found in the literature: just to give an example, we point to the work of Chen, Matano, and Vénon [10], where a *strongly order-preserving* semiflow is used to construct an entire orbit connecting two distinct solutions of a certain equation on the circle, which in turn allows the authors to obtain a singular solution of an equation of the form $\Delta u = |u|^{q-1}u$ in $\mathbb{R}^2 \setminus \{0\}$, $1 < q < 3$, and its behavior near the origin and infinity is characterized in terms of the foregoing two solutions connected by the entire orbit.

In this article our approach to spatial dynamics considers the use of the radial variable as the time-like variable, and the "cross-sections" are now concentric spheres. Although the idea of using the radial variable to take the role of time is not entirely new (see, e.g., [10, 29, 43, 45]), recently it has been explored in detail in the context of elliptic PDE by Beck et al. in [4, 5]. An interesting property of this approach is that the functional spaces involved consist of functions defined on spheres (or "sphere-like" bounded manifolds), so the study of the resulting equations could potentially be simpler. On the other hand, the abstract equation will in general depend on the time-like variable, which complicates its analysis even if one can construct a suitable invariant manifold. Under some structural assumptions and a suitable change of variables, the abstract equation does not depend on the time-like variable, which allows one to employ standard center manifold results. Although there are some center manifold results which may apply to more general settings than the one we consider here (e.g., [11, 12]), and it is likely that they could be used to construct new solutions, we do not make use of such results here: we expect that applying KAM theory to the equations resulting from such center manifold reductions would incur significant difficulties.

Among the challenges encountered when using a spatial dynamics approach to construct quasiperiodic solutions, a particularly relevant one is the verification of certain nondegeneracy conditions required to apply the KAM theory. In some settings it is possible to formulate such conditions explicitly in terms of the functions appearing in the original equation, but in general one often needs to restrict the scope of the results, for instance by restricting the number of frequencies or requiring the presence of a parameter in the equation, in order to obtain tangible hypotheses that can be shown to apply for certain classes of equations.

The rest of this paper is organized as follows. In Section 2 we provide some definitions and the statement of our main result, including the precise structure of the sought-after solutions. In Section 3 we apply the spatial dynamics approach to obtain a Hamiltonian structure for our equation, so that some previous results, based on KAM-type theorems, can be applied to obtain the desired solutions in Section 4.

2. Main result

In this section we introduce some terminology and provide the statement of our main result. Afterwards, we give an outline of the proof.

Throughout the paper, $C(X,Y)$ denotes the class of continuous functions $f: X \to Y$. Given a positive integer k, $C^k(X,Y)$ denotes the class of functions $f: X \to Y$ with continuous derivatives up to order k. Occasionally the spaces X and Y will be omitted from the notation if they are clear from the context. We write $C^k(X)$ for $C^k(X,\mathbb{R})$. We denote the unit sphere in $\mathbb{R}^N$ by $\mathbb{S}^{N-1}$, and the space $H^k(\mathbb{S}^{N-1})$ is the usual Sobolev space of square-integrable functions on $\mathbb{S}^{N-1}$ with weak derivatives up to the kth order. When needed, all the aforementioned spaces are equipped with the usual norms.

Given integers $n \geq 2$, $k \geq 1$, a vector $\omega = (\omega_1,\dots,\omega_n) \in \mathbb{R}^n$ is said to be *nonresonant up to order k* if

$$\omega \cdot \alpha \neq 0 \text{ for all } \alpha \in \mathbb{Z}^n \setminus \{0\} \text{ such that } |\alpha| \leq k. \tag{2.1}$$

(Here $|\alpha| = |\alpha_1| + \dots + |\alpha_n|$, and $\omega \cdot \alpha$ is the usual dot product.) If (2.1) holds for all $k = 1, 2, \dots$, we say that ω is *nonresonant*, or, equivalently, that the numbers $\omega_1,\dots,\omega_n$ are *rationally independent*.

A function $v : (\tau, \phi) \mapsto v(\tau, \phi) : \mathbb{R} \times \mathbb{S}^{N-1} \to \mathbb{R}$ is said to be *quasiperiodic* in τ if there exist an integer $n \geq 2$, a nonresonant vector $\omega^* = (\omega_1^*, \dots, \omega_n^*) \in \mathbb{R}^n$, and an injective function V defined on $\mathbb{T}^n$ (the n-dimensional torus) with values in the space of real-valued functions on $\mathbb{S}^{N-1}$ such that

$$v(\tau, \phi) = V(\omega_1^* \tau, \dots, \omega_n^* \tau)(\phi) \quad (\tau \in \mathbb{R},\ \phi \in \mathbb{S}^{N-1}). \tag{2.2}$$

The vector ω^* is called a *frequency vector* of v.

A function $u : \mathbb{R}^N \setminus \{0\} \mapsto \mathbb{R}$ is said to be *log-radially quasiperiodic* if there exist a constant a and a quasiperiodic function v (as in (2.2)) such that

$$u(r, \phi) = r^a v(\log r, \phi) \quad (r > 0,\ \phi \in \mathbb{S}^{N-1}). \tag{2.3}$$

We also say that ω^* is a frequency vector of u if ω^* is a frequency vector of v in the sense of the foregoing definition.

We emphasize that the nonresonance of the frequency vector is a part of our definitions. In particular, a quasiperiodic function is not periodic and, if it has some regularity properties, its image is dense in an n-dimensional manifold diffeomorphic to $\mathbb{T}^n$. As a consequence, a log-radially quasiperiodic function is also not periodic in $\log r$ (even if $a = 0$).

We now make precise the equation we study in this article. Denoting by $(r, \phi) \in (0, \infty) \times \mathbb{S}^{N-1}$ the spherical coordinates of $x \in \mathbb{R}^N \setminus \{0\}$, with $r = |x|$, we consider the following elliptic equation:

$$\Delta u + a_1(\phi; s) r^{-2} u + F(r, \phi, u; s) = 0, \quad x \in \mathbb{R}^N \setminus \{0\}, \tag{2.4}$$

where Δ is the Laplace operator in $\mathbb{R}^N$, $N \geq 2$, $s \approx 0$ is a parameter, and, setting

$$\mathcal{A} := (N-2)/2, \tag{2.5}$$

F takes the form

$$F(r, \phi, u; s) = r^{-(2+\mathcal{A})} f(\phi, r^{\mathcal{A}} u; s), \tag{2.6}$$

for

$$f(\phi, v; s) = a_2(\phi; s) v^2 + v^3 g(\phi, v; s). \tag{2.7}$$

Next, we provide some assumptions on the functions involved in equations (2.4) and (2.7). We assume that, for some $\delta > 0$ and for some integers K, m such that

$$K \geq 18, \quad m > \frac{N}{2}, \tag{2.8}$$

the functions a_1, a_2, and g satisfy the following hypotheses:

(A1) $a_1(\cdot; s) \in C^{m+1}(\mathbb{S}^{N-1})$ for each $s \in (-\delta, \delta)$, and the map $s \in (-\delta, \delta) \mapsto a_1(\cdot; s) \in C^{m+1}(\mathbb{S}^{N-1})$ is of class C^{K+1}.

(A2) $a_2(\cdot; s) \in C^{m+1}(\mathbb{S}^{N-1})$ for each $s \in (-\delta, \delta)$, the map $s \in (-\delta, \delta) \mapsto a_2(\cdot; s) \in C^{m+1}(\mathbb{S}^{N-1})$ is of class C^{K+1}; $g \in C^{K+m+4}(\mathbb{S}^{N-1} \times \mathbb{R} \times (-\delta, \delta))$, and for all $\chi > 0$ the function g is bounded on $\mathbb{S}^{N-1} \times [-\chi, \chi] \times [0, \delta)$ together with all its partial derivatives up to order $K + m + 4$.

Denote by $\Delta_{\mathbb{S}^{N-1}}$ the spherical Laplace operator on $\mathbb{S}^{N-1}$. The next hypotheses concern the Schrödinger operator $A_1(s) := -\Delta_{\mathbb{S}^{N-1}} - a_1(\phi; s)$, acting on $L^2(\mathbb{S}^{N-1})$ with domain $H^2(\mathbb{S}^{N-1})$.

(A3) For all $s \in [0, \delta)$, $A_1(s)$ has exactly two eigenvalues in $\left(-\infty, -\mathcal{A}^2\right]$. Denoting these two eigenvalues $\mu_1(s) < \mu_2(s)$, $\mu_2(s)$ is simple, and one has $\mu_2(s) < -\mathcal{A}^2$ for all $s \in (0, \delta)$ and $\mu_2(0) = -\mathcal{A}^2$.

(A4) Denoting
$$\vartheta_j(s) := \Big(\frac{\sqrt{|\mu_j(s)|} - \mathcal{A}}{\sqrt{|\mu_j(s)|} + \mathcal{A}}\Big)^{1/2}, \tag{2.9}$$
$j = 1, 2$, the vector
$$\tilde\omega(s) = (\tilde\omega_1(s), \tilde\omega_2(s)) := \big((\sqrt{|\mu_1(s)|} + \mathcal{A})\vartheta_1(s), (\sqrt{|\mu_2(s)|} + \mathcal{A})\vartheta_2(s)\big)$$
is nonresonant up to order K for all $s \in (0, \delta)$.

Hypotheses (A3) and (A4) are assumed in our main theorem, but in some of our results we consider more general versions of (A3) and (A4), namely:

(A3') There is an integer $n \geq 2$ such that for all $s \in (0, \delta)$, $A_1(s)$ has exactly n eigenvalues in $(-\infty, -\mathcal{A}^2)$, namely, $\mu_1(s) < \mu_2(s) < \cdots < \mu_n(s)$, all of which are simple. In addition, if $\mu_{n+1}(s)$ is the $(n+1)$-th eigenvalue of $A_1(s)$, one has $\mu_{n+1}(s) > -\mathcal{A}^2$ for all $s \in [0, \delta)$. ($\mu_n(0) = -\mathcal{A}^2$ is not required here.)

(A4') With ϑ_j as in (2.9), where now $j = 1, \dots, n$, the vector $\tilde\omega(s) = \big((\sqrt{|\mu_1(s)|} + \mathcal{A})\vartheta_1(s), \dots, (\sqrt{|\mu_n(s)|} + \mathcal{A})\vartheta_n(s)\big)$ is nonresonant up to order K for all $s \in (0, \delta)$, with K a positive integer satisfying
$$K \geq 6(n+1). \tag{2.10}$$

When hypotheses (A3') and (A4') are assumed in lieu of (A3) and (A4), the constant K in (A1), (A2) is also assumed to satisfy (2.10). Note that if $N = 2$, then $\mathcal{A} = 0$, $\vartheta_1(s) \equiv \cdots \equiv \vartheta_n(s) \equiv 1$, and $\tilde\omega(s) = \Big(\sqrt{|\mu_1(s)|}, \dots, \sqrt{|\mu_n(s)|}\Big)$.

For $s \in [0, \delta)$ and $j = 1, \dots, n$, we denote by $\varphi_j(\cdot; s)$ the eigenfunction of $A_1(s)$ associated with $\mu_j(s)$, normalized in the L^2-norm. This determines each φ_j uniquely up to a sign. Making a choice of sign for each j, the map $s \in [0, \delta) \mapsto \varphi_j \in H^2(\mathbb{S}^{N-1})$ is well defined and of class C^{K+1} [25]. Note that the exact choice of sign is inconsequential for our purposes.

Our last hypothesis concerns the coefficient a_2 and the eigenfunction φ_2 when $s = 0$:

(A5) One has
$$\int_{\mathbb{S}^{N-1}} a_2(\phi; 0)\varphi_2^3(\phi; 0)d\phi \neq 0.$$

Hypotheses (A1), (A2), (A3'), and (A4') with $m > N/2$ and $K \geq 6(n+1)$ are assumed throughout the paper. In our main theorem and its proof (Section 4), we take $n = 2$ and assume also that (A3) and (A5) hold.

Remark 2.1. (i) Since the eigenvalues of $A_1(s)$ are isolated in $\sigma(A_1(s))$, hypotheses (A3) and (A3') imply that there is $\gamma > -\mathcal{A}^2$ such that $(-\mathcal{A}^2, \gamma) \cap \sigma(A_1(s)) = \emptyset$ for all $s \in [0, \delta)$. Note that the operator $A_1(s)$ acts on functions defined on $\mathbb{S}^{N-1}$, so under our assumptions its spectrum consists only of eigenvalues.

(ii) Hypothesis (A1) implies that the eigenvalues $\mu_1(s)$, $\mu_2(s)$ in (A3) (or $\mu_1(s)$, $\dots, \mu_n(s)$ in (A3')) are functions of s of class C^{K+1} (see [25]). The simplicity of a finite set of eigenvalues of the Schrödinger operator $A_1(s)$ is a generic property (in a suitable sense) of the potential a_1, see [1]. Note, however, that the case of a_1 being constant in ϕ must be excluded, since the second eigenvalue of $-\Delta_{\mathbb{S}^{N-1}}$ is a multiple eigenvalue.

(iii) Note that if f is sufficiently smooth, then (2.7) is just a Taylor expansion of f around $v = 0$. The specific dependence on r in (2.6) is the most significant restriction we impose on F, and it is necessary for the applicability of standard center manifold results.

(iv) Condition (A4) holds automatically as long as $\delta > 0$ is sufficiently small: if $\mu_2(s)$ is sufficiently close to $-\mathcal{A}^2$, then one has $0 < K\tilde{\omega}_2(s) < \tilde{\omega}_1(s)$ for all $s \in (0, \delta)$, and (A4) can be easily verified using this fact. For (A4'), being a finite-order nonresonance condition, one can combine ideas from [1] with the scheme used in [39] to obtain that (A4') holds generically with respect to the potential a_1. Condition (A5) is obviously satisfied for "most" functions $a_2(\cdot; 0)$.

(v) Our hypotheses are for the most part analogous to some hypotheses in [38, 41]. This will allow us to use certain technical results from [38]. Hypothesis (A5) is specific to our approach to verify a certain nondegeneracy condition, in which we use Arnold's condition. There are other conditions used in KAM theory, such as Kolmogorov's or Bruno's conditions. In the presence of parameters other conditions can be used, see, e.g., [8, 47]. In principle any condition in a KAM-type theorem which permits the perturbed Hamiltonian to have only finite differentiability should suffice for our purposes.

We can now state our main theorem.

Theorem 2.2. *Suppose that hypotheses* (A1)–(A5) *with K, m as in* (2.8) *are satisfied. Then the following statements are valid, possibly after making $\delta > 0$ smaller, for each $s \in (0, \delta)$. There exists a solution $u = u(r, \phi)$ of equation* (2.4) *such that u is log-radially quasiperiodic. In fact, there is an uncountable family of such solutions, their frequency vectors forming an uncountable subset of $\mathbb{R}^2$.*

Remark 2.3. (i) For technical reasons (the verification of a nondegeneracy relation), in this theorem we need the parameter $s > 0$ to be sufficiently small and the number of frequencies to be restricted to $n = 2$. Below, we include a theorem – see Theorem 4.1 – where, assuming (A1), (A2), (A3'), and (A4'), we give a different sufficient condition for the existence of log-radially quasiperiodic solutions of (2.4) with any given number of frequencies and for a fixed value of s. Unlike (A5), that condition is rather implicit, and in general we are unable to formulate it as a specific condition on a_1, a_2.

(ii) We have taken a_1 and f (cf. (2.4) and (2.7), respectively) depending on $\phi \in \mathbb{S}^{N-1}$ for the sake of simplicity, but one could actually consider other "spherical-like" coordinate systems. For instance, if M is a sufficiently smooth manifold enclosing a star-shaped domain with respect to the origin, then one could consider a_1, f, and the sought-after solutions as functions depending on $r > 0$, $\phi \in M$, and our statements can be easily modified to apply in this new setting. The simplicity of the eigenvalues of $A_1(s)$ and hypothesis (A4) should also be generic in a suitable sense, again by arguments from [1].

(iii) The specific dependence of (2.4) in r allows us to apply standard center manifold results, see, e.g., [24, 51]. Such results are well suited to our approach because the resulting reduced equation inherits the Hamiltonian structure of the original equation. There are center manifold theorems for equations where the linear part of the equation is allowed to be non-autonomous, e.g., [11, 12, 45]. Such theorems may apply to a broader set of equations, and it is an interesting question, which we do not address in this article, whether such a reduction could be used to construct new solutions of equations of the form (2.4).

The proof of Theorem 2.2 follows a general scheme from [38, 41]. We express (2.4) in an abstract form and, after a reparametrization – where the logarithm of the radial variable takes the role of time, we apply a center manifold theorem. The resulting equation (the "reduced equation") is endowed with a Hamiltonian structure. After some transformations, the resulting Hamiltonian system is put in a form appropriate for some results from [41] to be applied, yielding quasiperiodic solutions of the abstract equation. These solutions correspond, in turn, to log-radially quasiperiodic solutions of (2.4).

3. Hamiltonian setting

To a significant extent, this section uses results from [38, 41], with changes to account for the setting of the present article. We first write equation (2.4) in abstract form, then apply a center manifold reduction, and endow the resulting equation with a Hamiltonian structure, which will be transformed to a form suitable for an application of a KAM-type theorem. Throughout this section we assume that hypotheses (A1), (A2), (A3'), and (A4') hold with $m > N/2$ and $K \geq 6(n+1)$.

To write (2.4) in abstract form, with $\log r$ taking the role of time, we start by recalling that

$$\Delta u = u_{rr} + \frac{N-1}{r} u_r + \frac{1}{r^2}\Delta_{\mathbb{S}^{N-1}} u,$$

where $\Delta_{\mathbb{S}^{N-1}}$ is the spherical Laplace operator on $\mathbb{S}^{N-1}$, the unit sphere in $\mathbb{R}^N$. Let F be as in (2.6), and consider the Nemytskii operator $\mathcal{F} : (0,\infty) \times H^{m+2}(\mathbb{S}^{N-1}) \times (-\delta,\delta) \to H^{m+1}(\mathbb{S}^{N-1})$ given by

$$\mathcal{F}(r,u;s)(\phi) = F(r,\phi,u(\phi);s) \quad (\phi \in \mathbb{S}^{N-1}).$$

This map is well defined and of class C^{K+1} in u. This fact can be proven using that $m > N/2$ (so $H^m(\mathbb{S}^{N-1})$ is a Banach algebra) and arguments from [49] or [38, Theorem A.1(b)].

For $t > 0$ and $s \in (-\delta,\delta)$, let

$$\begin{gathered} u_1(t;s)(\phi) = u(t,\phi;s), \\ u_2(t;s)(\phi) = \frac{\partial u}{\partial r}(t,\phi;s) \quad (\phi \in \mathbb{S}^{N-1}). \end{gathered}$$

Here we use $t > 0$ to emphasize that the radial variable r now takes the role of time. Equation (2.4) can thus be written in the form

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -t^{-2}\Delta_{\mathbb{S}^{N-1}} - t^{-2}a_1(\phi;s) & -(N-1)t^{-1} \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} 0 \\ \mathcal{F}(t,u_1;s) \end{pmatrix}, \tag{3.1}$$

for $t > 0$.

Following [4, Section 2], we consider the reparametrization $\tau = \log t$, and the functions

$$
\begin{gathered}
\tilde{u}_1(\tau) = e^{\mathcal{A}\tau} u_1(e^\tau), \\
\tilde{u}_2(\tau) = e^{(1+\mathcal{A})\tau} u_2(e^\tau), \\
\tilde{f}(\tilde{u}_1)(\phi) = e^{(2+\mathcal{A})\tau} \mathcal{F}(e^\tau, u_1(e^\tau))(\phi) = f(\phi, \tilde{u}_1),
\end{gathered} \tag{3.2}
$$

defined for $\tau \in \mathbb{R}$. The last equality in the third line of (3.2) is obtained using (2.6). Here $\mathcal{A} = (N-2)/2$, as in (2.5), and $\tilde{f} : H^{m+2}(\mathbb{S}^{N-1}) \times (-\delta, \delta) \to H^{m+1}(\mathbb{S}^{N-1})$ is the Nemytskii operator associated to f. From the regularity of $\mathcal{F}$ it follows that $\tilde{f}$ is of class C^K. Note that $\tilde{u}_1$, $\tilde{u}_2$, and $\tilde{f}$ all depend on the parameter s, but for the sake of notational simplicity we will drop that dependence from the notation when not needed (this will also apply to a_1 and other functions involving s). Note also that $\tilde{f}$ does not explicitly depend on τ.

Substituting (3.2) into (3.1), and expressing the system in terms of τ, we obtain

$$
\frac{d}{d\tau}\begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \begin{bmatrix} \mathcal{A} & 1 \\ -\Delta_{\mathbb{S}^{N-1}} - a_1(\phi) & -\mathcal{A} \end{bmatrix} \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} - \begin{pmatrix} 0 \\ \tilde{f}(\phi, \tilde{u}_1) \end{pmatrix}, \quad \tau \in \mathbb{R}. \tag{3.3}
$$

Denote

$$
\begin{gathered}
\tilde{u} = (\tilde{u}_1, \tilde{u}_2), \\
A_1(s) = -\Delta_{\mathbb{S}^{N-1}} - a_1(\cdot), \\
A(s) = \begin{bmatrix} \mathcal{A} & 1 \\ A_1(s) & -\mathcal{A} \end{bmatrix}, \\
R(\tilde{u}_1, \tilde{u}_2; s) = \begin{pmatrix} 0 \\ -\tilde{f}(\cdot, \tilde{u}_1) \end{pmatrix},
\end{gathered}
$$

so (3.3) becomes

$$
\frac{d}{d\tau}\tilde{u} = A(s)\tilde{u} + R(\tilde{u}; s), \quad \tau \in \mathbb{R}. \tag{3.4}
$$

Here, for each $s \in (-\delta, \delta)$, $A(s)$ is considered as an operator on the space $X := H^{m+1}(\mathbb{S}^{N-1}) \times H^m(\mathbb{S}^{N-1})$ and domain $D(A(s)) = Z := H^{m+2}(\mathbb{S}^{N-1}) \times H^{m+1}(\mathbb{S}^{N-1})$, and R as a C^{K+1}-map from $Z \times (-\delta, \delta)$ to Z. The concept of a solution of (3.4) on an interval $\mathcal{I}$ is as in [24, 51]: it is a function in $C^1(\mathcal{I}, X) \cap C(\mathcal{I}, Z)$ satisfying (3.4).

Given $s \in [0, \delta)$, to find the spectrum of $A(s)$ we consider the eigenvalue problem

$$
A(s)(\tilde{u}_1, \tilde{u}_2)^T = \nu(s)(\tilde{u}_1, \tilde{u}_2)^T,
$$

where the sought-after eigenvalues $\nu(s)$ depend on s. Using the definition of $A(s)$, this equation can be expanded as follows:

$$
\begin{gathered}
\mathcal{A}\tilde{u}_1 + \tilde{u}_2 = \nu(s)\tilde{u}_1 \\
A_1(s)\tilde{u}_1 - \mathcal{A}\tilde{u}_2 = \nu(s)\tilde{u}_2.
\end{gathered}
$$

Eliminate $\tilde{u}_2$ from the system to find

$$
A_1(s)\tilde{u}_1 = (\nu(s)^2 - \mathcal{A}^2)\tilde{u}_1;
$$

i.e., $\nu(s)$ is an eigenvalue of $A(s)$ if and only if $\nu(s)^2 - \mathcal{A}^2$ is an eigenvalue of $A_1(s)$. Denoting the eigenvalues of $A_1(s)$ as $\mu_\ell(s)$, $\ell = 1, 2, \dots$ in an increasing manner we find

$$
\nu_\ell^\pm(s) = \pm\sqrt{\mu_\ell(s) + \mathcal{A}^2}.
$$

Using (A3'), we find that $\nu_\ell^\pm(s) \in i\mathbb{R}$ (the imaginary axis) for $\ell = 1, \dots, n$ and $s \in [0, \delta)$, while there is some positive constant c such that $\nu_\ell^\pm(s) \in \mathbb{R} \setminus (-c, c)$ for all $s \in [0, \delta)$ and $\ell \geq n + 1$. Note also that, for each $s \in [0, \delta)$, the eigenvalues lying on the imaginary axis are all simple.

For $s \in [0, \delta)$, let $\varphi_j(\cdot; s)$, $j = 1, \dots, n$, be the eigenfunction of $A_1(s)$ corresponding to μ_j as introduced in Section 2 – in particular, owing to (A3'), the maps $s \mapsto \varphi_j(\cdot; s)$, $j = 1, \dots, n$ are well defined. By elliptic regularity, (A1) implies that $\varphi_j(\cdot; s) \in H^{m+2}(\mathbb{S}^{N-1})$, for $j = 1, \dots, n$ and $s \in [0, \delta)$. Moreover, by [25] and the regularity of μ_j with respect to s (cf. Remark 2.1(ii)), the maps $s \mapsto \varphi_j(\cdot; s)$ are of class C^{K+1} as $H^{m+2}(\mathbb{S}^{N-1})$-valued functions of s.

We define the space

$$X_c(s) := \big\{(h, \tilde{h})^T : h, \tilde{h} \in \operatorname{span}\{\varphi_1(\cdot; s), \dots, \varphi_n(\cdot; s)\}\big\} \subset Z,$$

the orthogonal projection operator $\Pi(s) : L^2(\mathbb{S}^{N-1}) \to \operatorname{span}\{\varphi_1(\cdot; s), \dots, \varphi_n(\cdot; s)\}$, and let $P_c(s) : X \to X_c(s)$ be given by $P_c(s)(v_1, v_2) = (\Pi(s)v_1, \Pi(s)v_2)$. This operator is the spectral projection for the operator $A(s)$ associated with the spectral set $\{\nu_\ell^\pm(s) : \ell = 1, \dots, n\}$, cf. [38, Section 3.2], and it is well defined, since the rest of the spectrum of $A(s)$ is at a positive distance (independent of s) from the imaginary axis. Due to (A1), the map $s \mapsto P_c(s)$ is of class C^{K+1} from $s \in [0, \delta)$ to the class of linear bounded operators on X; moreover, the smoothness of the maps $s \mapsto \varphi_j(\cdot; s)$ implies that $s \mapsto P_c(s)$ is of class C^{K+1} as a map on $[0, \delta)$ with values on the class of linear bounded operators from X to Z.

Also we define $P_h(s) = I_X - P_c(s)$, I_X being the identity map on X, and, for $j = 1, \dots, n$,

$$\psi_j(\cdot; s) = (\varphi_j(\cdot; s), 0)^T, \quad \zeta_j(\cdot; s) = (0, \varphi_j(\cdot; s))^T. \tag{3.5}$$

A basis of $X_c(s)$ is given by

$$\mathscr{B}(s) := \{\psi_1(\cdot; s), \dots, \psi_n(\cdot; s), \zeta_1(\cdot; s), \dots, \zeta_n(\cdot; s)\}.$$

For $z \in X_c(s)$, we denote by $\{z\}_{\mathscr{B}}$ the coordinates of z with respect to the basis $\mathscr{B}(s)$. Denote further

$$\begin{aligned} \psi(s) &:= (\psi_1(\cdot; s), \dots, \psi_n(\cdot; s)), \\ \zeta(s) &:= (\zeta_1(\cdot; s), \dots, \zeta_n(\cdot; s)). \end{aligned} \tag{3.6}$$

Proposition 3.1. *Using the above notation the following statement is valid, possibly after making $\delta > 0$ smaller. There exist a map $\sigma : (\xi, \eta; s) \in \mathbb{R}^{2n} \times [0, \delta) \mapsto \sigma(\xi, \eta; s) \in Z$ of class C^{K+1} and a neighborhood $\mathscr{N}$ of 0 in Z such that for each $s \in [0, \delta)$ one has*

$$\sigma(\xi, \eta; s) \in P_h(s)Z \quad ((\xi, \eta) \in \mathbb{R}^{2n}), \tag{3.7}$$

$$\sigma(0, 0; s) = 0, \quad D_{(\xi,\eta)}\sigma(0, 0; s) = 0, \tag{3.8}$$

and the manifold

$$W_c(s) = \{\xi \cdot \psi(s) + \eta \cdot \zeta(s) + \sigma(\xi, \eta; s) : (\xi, \eta) = (\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n) \in \mathbb{R}^{2n}\} \subset Z$$

has the following properties:

(a) *If $\tilde{u}(\tau)$ is a solution of (3.4) on $\mathcal{I} = \mathbb{R}$ and $\tilde{u}(\tau) \in \mathscr{N}$ for all $\tau \in \mathbb{R}$, then $\tilde{u}(\tau) \in W_c(s)$ for all $\tau \in \mathbb{R}$; that is, $W_c(s)$ contains the orbit of each solution of (3.4) which stays in $\mathscr{N}$ for all $\tau \in \mathbb{R}$.*

(b) *If $z : \mathbb{R} \to X_c(s)$ is a solution of the equation*
$$\frac{dz}{d\tau} = A(s)\big|_{X_c(s)} z + P_c(s)R(z + \sigma(\{z\}_{\mathscr{B}}; s); s) \tag{3.9}$$
on some interval $\mathcal{I}$, and $\tilde{u}(\tau) := z(\tau) + \sigma(\{z(\tau)\}_{\mathscr{B}}; s) \in \mathscr{N}$ for all $\tau \in \mathcal{I}$, then $\tilde{u} : \mathcal{I} \to Z$ is a solution of (3.4) on $\mathcal{I}$.

Moreover, σ satisfies the following relation:

(c) *If $2 \le \ell \le K$ is an integer, then $\sigma(\{\tilde{u}\}_{\mathscr{B}}; s) = \mathcal{O}(\|\tilde{u}\|^\ell)$ as $\tilde{u} \to 0$ whenever $s \in [0, \delta)$ is such that $R(\tilde{u}; s) = \mathcal{O}(\|\tilde{u}\|^\ell)$ as $\tilde{u} \to 0$.*

From now on, the function σ is called the *reduction function*, $W_c(s)$ is the *center manifold*, and equation (3.9) is the *reduced equation*. In the sequel it will be convenient to write $\sigma = (\sigma_1, \sigma_2)$, where $\sigma_1 \in H^{m+2}(\mathbb{S}^{N-1})$, $\sigma_2 \in H^{m+1}(\mathbb{S}^{N-1})$.

The proof of Proposition 3.1 can be found in [41]. For the most part, the conclusions of Proposition 3.1 are standard conclusions of center manifold theorems found in the literature [24, 51], but some additional work is needed to obtain the desired regularity in s, since the parameter s appears in the linear term (albeit only in the bounded part of the linear term). Note that in [41] the space Z was taken to be $H^{m+2}(\mathbb{R}^N) \times H^{m+1}(\mathbb{R}^N)$, but the specifics of the space (other than the fact that it is a Hilbert space) are not relevant in the proofs. Similarly, the regularity assumptions for the nonlinear term R rely on the regularity of the Nemytskii operator $\tilde{f}$, discussed above, so all these results apply in the present setting.

Remark 3.2. (i) In our case statement (c) of Proposition 3.1 applies with $\ell = 2$, so $\sigma(\{\tilde{u}\}_{\mathscr{B}}; s) = \mathcal{O}(\|\tilde{u}\|^2)$ as $\tilde{u} \to 0$, or, equivalently, $\sigma(\xi, \eta; s) = \mathcal{O}(|(\xi, \eta)|^2)$ as $(\xi, \eta) \to (0, 0)$ uniformly in s.

(ii) The components σ_1 and σ_2 of σ take values in the orthogonal complement (with respect to the L^2-inner product) of $\operatorname{span}\{\varphi_1(\cdot; s), \dots, \varphi_n(\cdot; s)\}$. In addition, $\operatorname{span}\{\varphi_1(\cdot; s), \dots, \varphi_n(\cdot; s)\}$ and its orthogonal complement are invariant under the operator $A_1(s)$. These facts will be used below.

To endow the reduced equation corresponding to (3.4) with a Hamiltonian structure, we first study the (formal) Hamiltonian structure of (3.4), since this structure is inherited (in a precise sense) by the reduced equation [32].

Let $\mathscr{F}(\phi, u; s) := \int_0^u f(\phi, w; s)dw$ for $s \in [0, \delta)$, $\phi \in \mathbb{S}^{N-1}$ (f is as in (2.7)), and, for $(\tilde{u}_1, \tilde{u}_2) \in Z$,
$$H(\tilde{u}_1, \tilde{u}_2; s) := \int_{\mathbb{S}^{N-1}} \Big(\mathcal{A}\tilde{u}_1\tilde{u}_2 + \frac{1}{2}\tilde{u}_2^2 - \frac{1}{2}|\nabla \tilde{u}_1|^2 + \frac{1}{2}a_1(\phi; s)\tilde{u}_1^2 + \mathscr{F}(\tilde{u}_1) \Big) d\phi, \tag{3.10}$$
where ∇ stands for the spherical gradient.

Equation (3.3) has a formal Hamiltonian structure with respect to the functional H and the canonical symplectic structure on $L^2(\mathbb{S}^{N-1}) \times L^2(\mathbb{S}^{N-1})$. Its restriction to the center manifold yields the Hamiltonian of the reduced equation. More precisely, let
$$\Phi(\xi, \eta; s) = H\big(\xi \cdot \varphi(s) + \sigma_1(\xi, \eta; s), \eta \cdot \varphi(s) + \sigma_2(\xi, \eta; s); s\big), \tag{3.11}$$
where $\varphi(s) = (\varphi_1(s), \dots, \varphi_n(s))$, $\xi \cdot \varphi(s) = \xi_1\varphi_1(s) + \dots + \xi_n\varphi_n(s)$ and similarly for $\eta \cdot \varphi(s)$. Then Φ is a map from $\mathbb{R}^{2n} \times [0, \delta)$ to $\mathbb{R}$, and (3.9) is the Hamiltonian system with respect to the Hamiltonian Φ and a certain symplectic structure defined in a neighborhood of $(0, 0) \in \mathbb{R}^{2n}$. This can be proved using general statements in [32], but in [38, 41] we can find results which contain additional information regarding

the dependence of Φ on (ξ,η) and s. The computations performed in those papers apply here as well, thus, aside from stating the relevant equations to account for the differences in our current setting, we will omit most of the proofs, which are quite technical and do not require any meaningful changes to be valid in the present setting. The following result, also used in [38, 41] will be relevant later on.

Lemma 3.3. *The quadratic and cubic terms (in (ξ,η)) of Φ are independent of the reduction function σ.*

Proof. Noting that $\sigma=(\sigma_1,\sigma_2)$ is of order $\mathcal{O}(|(\xi,\eta)|^2)$ as $(\xi,\eta)\to(0,0)$ (cf. Remark 3.2(i)), we see that the quadratic terms (in (ξ,η)) of Φ do not involve the function σ. In order to study the terms of degree 3, we note first that

$$\int_{\mathbb{S}^{N-1}}\Big(-\frac{1}{2}|\nabla\tilde u_1|^2+\frac{1}{2}a_1(\phi;s)\tilde u_1^2\Big)d\phi=\frac{1}{2}\int_{\mathbb{S}^{N-1}}(\Delta_{\mathbb{S}^{N-1}}\tilde u_1+a_1\tilde u_1)\tilde u_1\ d\phi.$$

Recalling that $\Delta_{\mathbb{S}^{N-1}}\varphi_j+a_1\varphi_j=-A_1(s)\varphi_j=-\mu_j\varphi_j$, we notice that the cubic terms resulting from taking $\tilde u_1=\xi\cdot\varphi(s)+\sigma_1(\xi,\eta;s)$ and $\tilde u_2=\eta\cdot\varphi(s)+\sigma_2(\xi,\eta;s)$ as in (3.11) are terms that either do not involve σ, or of the form

$$\int_{\mathbb{S}^{N-1}}\Big(\sum_{j=1}^{n}(a_j\xi_j\varphi_j+b_j\eta_j\varphi_j)\Big)G(\xi,\eta)\ d\phi, \tag{3.12}$$

where a_j, b_j are some constants depending on s, independent of (ξ,η) and ϕ, and G is equal to either $\sigma_\ell(\xi,\eta)$, $\ell=1,2$, or $-A_1(s)\sigma_1(\xi,\eta)$. In either case, G and φ_j are orthogonal by Remark 3.2(ii). We thus conclude that the integral in (3.12) vanishes, whence (3.11) does not contain any nonzero cubic terms involving σ. $\square$

The Hamiltonian system with functional Φ can be successively transformed by performing three coordinate changes:

a Darboux transformation, normal form transformation, and action-angle variables. (3.13)

By the first change of coordinates, we achieve that the transformed system is Hamiltonian with respect to the standard symplectic form on $\mathbb{R}^{2n}$ (and the transformed Hamiltonian functional). The existence of such a local transformation is guaranteed by the Darboux theorem, but we need some more precise statements found in [38], which provide additional details on the dependence of the transformation on the parameter s and on the coordinates (ξ,η). In particular, the Darboux transformation can be chosen as the sum of the identity map (on $\mathbb{R}^{2n}$) and terms of order $\mathcal{O}(|(\xi,\eta)|^3)$, with the cubic terms having coefficients of class C^K in s. This implies that the Darboux transformation does not change the quadratic or cubic terms of Φ, but it may alter terms of degree 4 and higher.

In the new coordinates (still denoted (ξ,η)) resulting from the aforementioned Darboux transformation, the Hamiltonian takes the following form for $(\xi,\eta)\approx(0,0)$:

$$\begin{aligned}\Phi(\xi,\eta;s)&=\frac{1}{2}\sum_{j=1}^{n}\big(-\mu_j(s)\xi_j^2+2\mathcal{A}\xi_j\eta_j+\eta_j^2\big)\\&\quad+\frac{1}{3}\int_{\mathbb{S}^{N-1}}a_2(\phi;s)(\xi\cdot\varphi(\phi;s))^3\ d\phi+\Phi_4(\xi,\eta;s)+\Phi'(\xi,\eta;s).\end{aligned}\tag{3.14}$$

Here Φ_4 is a homogeneous polynomial in (ξ,η) of degree 4 whose coefficients are of class C^K in $s\in[0,\delta)$ (in particular, their C^K-norm is bounded), and Φ' is a function of class C^K in all its arguments and of order $\mathcal{O}(|(\xi,\eta)|^5)$ as $(\xi,\eta)\to(0,0)$. Note that, thanks to Lemma 3.3 and our choice of Darboux transformation, the quadratic and cubic terms of Φ are explicitly known, as the reduction function σ and the terms introduced by the Darboux transformation are present only in terms of degree 4 and higher. Also, all the changes of variables we consider below will be canonical changes, that is, the (canonical) symplectic structure will be preserved.

For $j=1,\dots,n$, denote $\omega_j=\sqrt{|\mu_j|}$ (ω_j depends on s, but for the sake of notational clarity we omit the dependence), and consider the change of coordinates

$$\xi_j=(\omega_j)^{-1/2}\xi'_j,\quad \eta_j=(\omega_j)^{1/2}\eta'_j,$$

so the Hamiltonian Φ becomes

$$\begin{aligned}\Phi(\xi',\eta')&=\frac12\sum_{j=1}^n\left(\omega_j\xi_j'^2+2\mathcal{A}\xi'_j\eta'_j+\omega_j\eta_j'^2\right)\\&\quad+\frac13\int_{\mathbb{S}^{N-1}}a_2(\phi)(\xi\cdot\varphi(\phi))^3\,d\phi+\Phi_4(\xi',\eta')+\Phi'(\xi',\eta').\end{aligned}\tag{3.15}$$

Here $\Phi(\xi',\eta')$ stands for $\Phi(\xi(\xi'),\eta(\eta'))$ (same for Φ_4 and Φ'). For the time being we postpone expanding $\xi\cdot\varphi$ in terms of ξ'.

Next, we diagonalize the quadratic terms of Φ. If $N=2$, then $\mathcal{A}=0$, and nothing needs to be done. If $N\geq 3$, define ϑ_j as in (2.9) and consider the (canonical) transformation

$$\tilde\xi_j=\frac{\sqrt{\vartheta_j}}{\sqrt2}(\xi'_j-\eta'_j),\quad \tilde\eta_j=\frac{1}{\sqrt2\sqrt{\vartheta_j}}(\xi'_j+\eta'_j).$$

In the new coordinates,

$$\begin{aligned}&\Phi(\tilde\xi,\tilde\eta)\\&=\sum_{j=1}^n(\omega_j+\mathcal{A})\vartheta_j\Big(\frac{\tilde\xi_j^2+\tilde\eta_j^2}{2}\Big)\\&\quad+\frac13\int_{\mathbb{S}^{N-1}}a_2(\phi)\Big[\sum_{j=1}^n\frac{1}{\sqrt2\sqrt{\omega_j}}\Big(\frac{1}{\sqrt{\vartheta_j}}\tilde\xi_j+\sqrt{\vartheta_j}\tilde\eta_j\Big)\varphi_j(\phi)\Big]^3d\phi+\text{h.o.t.},\end{aligned}\tag{3.16}$$

where h.o.t. stands for terms of order $\mathcal{O}(|(\tilde\xi,\tilde\eta)|^4)$ as $|(\tilde\xi,\tilde\eta)|\to0$, and the term in brackets is the expansion of $(\xi\cdot\varphi(\phi))$ from (3.15), now written in terms of $(\tilde\xi,\tilde\eta)$.

The Hamiltonian Φ in (3.16) (or in (3.15) if $N=2$) has thus been written in a suitable form so that the second transformation in (3.13) can be performed: for $s>0$, the Hamiltonian $\Phi(\cdot,\cdot;s)$ is transformed to its normal form up to order $2k_B+1$, where $k_B:=[K/2]-1$, $[K/2]$ being the integer part of $K/2$. More precisely, near $(0,0)$ there is a canonical coordinate transformation such that in the new coordinates $(\bar\xi,\bar\eta)$ the Hamiltonian can be written as follows. Let $(\bar\xi,\bar\eta)=(\bar\xi_1,\dots,\bar\xi_n,\bar\eta_1,\dots,\bar\eta_n)$,

$$I_j=\frac12(\bar\xi_j^2+\bar\eta_j^2)\quad(j=1,\dots,n),\tag{3.17}$$

and $I=(I_1,\dots,I_n)$. Then

$$\Phi(\bar\xi,\bar\eta;s)=\tilde\omega(s)\cdot I+\Phi_0(I;s)+\Phi_1(\bar\xi,\bar\eta;s),\tag{3.18}$$

where

$$\tilde\omega(s) = (\tilde\omega_1(s), \dots, \tilde\omega_n(s)) := \big((\omega_1(s) + \mathcal{A})\vartheta_1(s), \dots, (\omega_n(s) + \mathcal{A})\vartheta_n(s)\big), \quad (3.19)$$

Φ_0 is a polynomial in I of degree at most k_B, and Φ_1 a C^K function of order $\mathcal{O}(|(\bar\xi, \bar\eta)|^{2k_B+2})$ as $(\bar\xi, \bar\eta) \to (0,0)$. (Note that if $N = 2$, then $\tilde\omega_j = \omega_j$ for $j = 1, \dots, n$.) The polynomial Φ_0 is of the form

$$\Phi_0(I; s) = \frac{1}{2} I \cdot M(s) I + \hat P(I; s), \quad (3.20)$$

where, for $s \in (0, \delta)$, $M(s)$ is an $n \times n$ matrix and $\hat P(I; s)$ a polynomial in I (of degree at most k_B) with no constant, linear, or quadratic terms. The entries of $M(s)$ and the coefficients of $\hat P(\cdot; s)$ are of class C^K in s.

For the final transformation in (3.13), we introduce the *action-angle* variables $I = (I_1, \dots, I_n) \in \mathbb{R}^n$, $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{T}^n$ by

$$(\bar\xi_j, \bar\eta_j) = \sqrt{2I_j}(\cos\theta_j, \sin\theta_j).$$

The change of coordinates from $(\bar\xi_j, \bar\eta_j)$ to (θ, I) is defined in regions where $I_j = (\bar\xi_j^2 + \bar\eta_j^2)/2 > 0$ for all $j \in \{1, \dots, n\}$, and it is well known that this transformation is canonical. In these coordinates, Φ looks as follows:

$$\Phi(\theta, I; s) = \tilde\omega(s) \cdot I + \Phi_0(I; s) + \Phi_1(\theta, I; s). \quad (3.21)$$

($\Phi(\theta, I; s)$ actually stands for the function $\Phi(\bar\xi(\theta, I), \bar\eta(\theta, I); s)$, and similarly for Φ_0, Φ_1.) Thus, the Hamiltonian Φ is the sum of an integrable Hamiltonian (the first two terms on the right hand side of (3.21)) and a "perturbation" (the last term in (3.21)). This is a form suitable for an application of a KAM-type theorem.

4. Proof of Theorem 2.2

Once the Hamiltonian Φ of the reduced equation corresponding to the abstract equation (3.4) has been rewritten in the form (3.21), one can use results from [41] to obtain the existence of quasiperiodic solutions of (3.4). Using a theorem from [4], those solutions correspond to log-radially quasiperiodic solutions of our original equation (2.4).

We first consider the more general case of log-radially quasiperiodic solutions of (2.4) with n frequencies. In order to do so, we need the following additional hypothesis on Φ, the transformed Hamiltonian of the reduced equation as in (3.21):

(A6) Consider the $(n+1) \times (n+1)$ matrix

$$\mathcal{M}(s) := \begin{bmatrix} D^2\Phi_0(0; s) & \tilde\omega(s) \\ \tilde\omega^T(s) & 0 \end{bmatrix}. \quad (4.1)$$

Then at least one of the matrices $D^2\Phi_0(0; s)$ and $\mathcal{M}(s)$ is nonsingular.

Theorem 4.1. *Assume that hypotheses* (A1), (A2), (A3'), (A4') *are satisfied, and that* (A6) *holds for some fixed* $s \in (0, \delta)$. *Then there exists a solution* $u = u(r, \phi)$ *of equation* (2.4) *such that* u *is log-radially quasiperiodic with a (nonresonant) frequency vector in* $\mathbb{R}^n$. *Moreover, there is an uncountable family of such log-radially quasiperiodic solutions, their frequency vectors forming an uncountable subset of* $\mathbb{R}^n$.

The proof of this theorem consists of two parts. The first step is obtaining a pair of quasiperiodic functions $(\tilde{u}_1, \tilde{u}_2)$ which satisfy (3.4), that is, $\tilde{u}_1$ and $\tilde{u}_2$ are such that the maps $(\tau, \phi) \mapsto \tilde{u}_j(\tau)(\phi)$, $j = 1, 2$, are quasiperiodic in the sense of the definition in Section 2 (cf. equation (2.2)). This step is analogous to a theorem in [41]. Once such a pair is obtained, we need to establish that there is a solution u of (2.4) corresponding to the pair $(\tilde{u}_1, \tilde{u}_2)$. In order to do so, we make use of the following result contained in [4, Theorem 3.6]:

Theorem 4.2. *Suppose* $0 < T < \infty$. *If* (u_1, u_2) *is a solution of* (3.1) *on* $(0, T)$ *(for a fixed value of s), then there exists a weak solution* u *of* (2.4) *on* $B(0, T) \setminus \{0\}$ *such that* $u(t, \cdot) = u_1(t)$, $\frac{\partial u}{\partial r}(t, \cdot) = u_2(t)$ *for each* $t \in (0, T)$.

The definition of weak solution used in [4] is as follows. Given $b > a > 0$, let $\Omega = \{x \in \mathbb{R}^N : a < |x| < b\}$. Then u is a weak solution of (2.4) on Ω if $u \in H^1(\Omega)$, $a_1(\phi; s)r^{-2}u + F(r, \phi, u; s) \in L^2(\Omega)$, and

$$\int_a^b \int_{\mathbb{S}^{N-1}} \nabla u \cdot \nabla v \, d\phi dr = \int_a^b \int_{\mathbb{S}^{N-1}} \big(a_1(\phi; s)r^{-2}u + F(r, \phi, u; s)\big) \, v \, d\phi dr$$

holds for all $v \in H_0^1(\Omega)$. (Here ∇ is the usual gradient.) In the case $\Omega = B(0, b) \setminus \{0\}$, we say u is a weak solution on Ω if u is a weak solution on $B(0, b) \setminus \overline{B(0, b')}$ for each $b' \in (0, b)$. By standard regularity arguments, if u is a weak solution of (2.4) on a domain away from the origin, then u is a classical solution as well.

Remark 4.3. The aforementioned theorem in [4] applies to a wider class of geometrical settings (see Hypothesis 3.1 in [4]), in which case the abstract formulation of a semilinear elliptic equation is more involved (cf. [4, Equation (14)]). In general it is to be expected that the linear part of the abstract formulation will not be autonomous, and that this will not be remedied by a change of variables such as (3.2). This would preclude the application of classical center manifold reductions as found in, say, [24, 51], where it is essential to have the linear part of the equation to be autonomous.

Proof of Theorem 4.1. One can follow the proof of [41, Theorem 4.4] to construct quasiperiodic solutions $(\tilde{u}_1, \tilde{u}_2)$ of (3.4), parametrized by their frequency vectors, which are nonresonant and form an uncountable subset of $\mathbb{R}^n$. Each pair $(\tilde{u}_1, \tilde{u}_2)$ corresponds to a solution (u_1, u_2) of (3.1) via (3.2) and the reparametrization $t = e^\tau$, which implies that u_1 and u_2 are log-radially quasiperiodic (with $a = -\mathcal{A}$ and $a = -\mathcal{A} - 1$, respectively). Using Theorem 4.2, there is a corresponding log-radially quasiperiodic solution u of (2.4) on domains of the form $\{x \in \mathbb{R}^N : 0 < |x| < T\}$ for any $T > 0$, which satisfies $u(r, \cdot) = u_1(r)$, $\frac{\partial u}{\partial r}(r, \cdot) = u_2(r)$. This allows us to define u on $\mathbb{R}^N \setminus \{0\}$. $\square$

Although we do not reproduce the proof of [41, Theorem 4.4] here, for the reader's convenience we provide a brief sketch. The Hamiltonian (3.21) can be seen as a near-integrable Hamiltonian, in the sense that if I is sufficiently small, then Φ is the sum of an analytic integrable Hamiltonian (namely, the first two terms in (3.21)) and a perturbation term Φ_1, which is of order $\mathcal{O}(|I|^{k_B+1})$ (k_B is the constant considered in the paragraph after (3.16)), so this term is small if the domain for I is sufficiently small. This is the standard setting for KAM-type results. In order to apply a KAM-type theorem, one usually requires a *Diophantine* condition and some *nondegeneracy* condition, the former condition being relatively easy to verify once it is shown that the latter holds. Hypothesis (A6) provides two options to verify a nondegeneracy condition: if $D^2\Phi_0$ is nonsingular (often referred to as Kolmogorov's condition), then a theorem by Pöschel [42] can be applied (as in [38]) to yield the existence of the desired quasiperiodic solutions for (3.4); if $\mathcal{M}$ is nonsingular (known as Arnold's condition), then a result from [7] allows one to apply the result in [42] to an auxiliary Hamiltonian, which, after a suitable rescaling, yields again the desired quasiperiodic solutions for (3.4).

Note that the Hamiltonian Φ in (3.21) takes the same form as the Hamiltonian in [41, Equation (3.25)]. Throughout the proof of [41, Theorem 4.4] the original elliptic equation and the abstract equation play no role whatsoever, which allows one to use the same arguments to obtain a solution of (3.4).

We can now prove Theorem 2.2. We henceforth fix $n = 2$ (the number of frequencies), and assume (A1)–(A5) hold. The proof relies on a careful study of the normal form procedure, similar to [41]. Recall that the Birkhoff normal form algorithm consists of successive transformations eliminating inessential terms of a given degree, which introduces new terms of higher degree, but leaving lower order terms unchanged. In our setting, the first transformation eliminates all cubic terms, introducing new terms of degree 4 and higher. The second transformation eliminates *nonresonant* terms of degree 4 and leaves the remaining *resonant* terms of degree 4 unchanged (see, e.g., [21] for a detailed discussion of the Birkhoff normal form algorithm). Careful computations allow us to study the asymptotic behavior of $\det \mathcal{M}(s)$ as $s \to 0$; more precisely, we determine which term of degree 4 (after the first transformation) grows at the fastest rate as $s \to 0$.

Proof of Theorem 2.2. If hypothesis (A6) holds for the Hamiltonian Φ in (3.21) for each $s \in (0, \delta)$, then the result is a direct consequence of Theorem 4.1. Therefore we will show that in our setting hypotheses (A3) and (A5) imply (A6) for each $s \in (0, \delta)$, where $\delta > 0$ is sufficiently small. In order to do this, we return to the Hamiltonian Φ as found in (3.16) (or (3.15) if $N = 2$). We recall that at this point the Hamiltonian has been written in a standard form suitable for the application of a Birkhoff normal form algorithm, as outlined in, e.g., [3, 21]. As before, for the sake of clarity we will drop the dependence in s from the notation whenever it does not play a relevant role.

We first assume $N \geq 3$. The cubic terms (in $(\tilde{\xi}, \tilde{\eta}) = (\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\eta}_1, \tilde{\eta}_2)$) of Φ can be written as

$$\Phi_3(\tilde{\xi}, \tilde{\eta}) = \sum_{j,k,\ell=1}^{2} \Theta(j,k,\ell)\tilde{\xi}_j\tilde{\xi}_k\tilde{\xi}_\ell + \Phi_3^r(\tilde{\xi}, \tilde{\eta}),$$

where

$$\Theta(j,k,\ell) = \frac{1}{3(\omega_j\omega_k\omega_\ell)^{1/2}} \frac{1}{2\sqrt{2}(\vartheta_j\vartheta_k\vartheta_\ell)^{1/2}} \int_{\mathbb{S}^{N-1}} a_2\varphi_j\varphi_k\varphi_\ell d\phi, \tag{4.2}$$

and

$$\Phi_3^r(\tilde{\xi}, \tilde{\eta}) = \sum_{j,k,\ell=1}^{2} \Theta(j,k,\ell) \left(3\vartheta_\ell\tilde{\xi}_j\tilde{\xi}_k\tilde{\eta}_\ell + 3\vartheta_k\vartheta_\ell\tilde{\xi}_j\tilde{\eta}_k\tilde{\eta}_\ell + \vartheta_j\vartheta_k\vartheta_\ell\tilde{\eta}_j\tilde{\eta}_k\tilde{\eta}_\ell\right),$$

i.e., Φ_3^r comprises all cubic terms of the Hamiltonian Φ involving at least one factor $\tilde{\eta}_1$ or $\tilde{\eta}_2$. As before, φ_j stands for the normalized eigenfunction of $-\Delta_{\mathbb{S}^{N-1}} - a_1$

associated to μ_j, as in Section 2, $\omega_j(s) = \sqrt{|\mu_j(s)|}$, and

$$\vartheta_j(s) = \Big(\frac{\omega_j(s) - \mathcal{A}}{\omega_j(s) + \mathcal{A}}\Big)^{1/2}$$

is as in (2.9), $j = 1, 2$.

Making $\delta > 0$ smaller if necessary, we have that, by (A3) and our assumption $N \geq 3$, ω_1 and ω_2 satisfy $\omega_1(s) > c_\delta > \omega_2(s) \geq \mathcal{A} > 0$ for all $s \in [0, \delta)$, where $c_\delta > \mathcal{A}$ is a constant depending on δ, but independent of s. We conclude that there is a constant $c > 0$ such that $\vartheta_1(s) \geq c > 0$ holds for all $s \in [0, \delta)$, while $\vartheta_2(s) \to 0+$ as $s \to 0$, this limit coming from the assumption $\mu_2(0) = -\mathcal{A}^2$ in (A3). Since the maps $s \in [0, \delta) \mapsto a_2(\cdot; s) \in C^{m+1}(\mathbb{S}^{N-1})$ and $s \in [0, \delta) \mapsto \varphi_j(\cdot; s) \in L^\infty(\mathbb{S}^{N-1})$ are continuous, the integral in (4.2) is bounded by a constant independent of s. The foregoing statements imply that

$$\Theta(j, k, \ell; s) = \mathcal{O}\Big(\vartheta_2^{-(j+k+\ell-3)/2}\Big) \quad (j, k, \ell \in \{1, 2\})$$

as $s \to 0$. In particular,

$$\begin{gathered} \Theta(2, 2, 2; s) = \mathcal{O}(\vartheta_2^{-3/2}), \\ \Theta(j, k, \ell; s) = \mathcal{O}(\vartheta_2^{-1}) \quad \text{if } (j, k, \ell) \neq (2, 2, 2). \end{gathered}$$

From the asymptotic behavior of $\Theta(j, k, \ell; s)$ we also conclude that all the coefficients in Φ_3^r are of order $\mathcal{O}(\vartheta_2^{-1})$ as $s \to 0$. The coefficients of the terms of degree 4 in $(\tilde{\xi}, \tilde{\eta})$ can be shown to be of order $\mathcal{O}(\vartheta_2^{-2})$ as $s \to 0$ by a similar argument.

One can now apply the Birkhoff normal form algorithm to eliminate all terms of degree 3 (in $(\tilde{\xi}, \tilde{\eta})$), which introduces new terms of degree 4 (and higher). As discussed above, the next transformation eliminates some terms of degree 4, while the remaining terms are unchanged. After the change of variables (3.17), one can study the asymptotic behavior of the nonresonant terms of degree 2 in $I = (I_1, I_2)$ as in [41, Lemma 5.4] to obtain

$$\begin{aligned} \Phi_0(I; s) = {} & \frac{C}{\omega_2^{3/2}(s)(\omega_2(s) + \mathcal{A})\vartheta_2^4(s)} \Big(\int_{\mathbb{S}^{N-1}} a_2(\phi; s)\varphi_2^3(\phi; s) d\phi\Big)^2 I_2^2 + \\ & + \tilde{\Phi}(I; s) + \text{h.o.t.}, \end{aligned}$$

where C is a positive constant independent of s, and $\tilde{\Phi}$, comprising all remaining quadratic terms (in I), has coefficients of order $\mathcal{O}(\vartheta_2^{-7/2})$ as $s \to 0$, while h.o.t. stands for terms of degree 3 and higher in I. Recalling that $N \geq 3$, so $\omega_2(s) \geq \mathcal{A} > 0$, we can prove that the matrix $\mathcal{M}(s)$, defined in (A6), is nonsingular for all $s \in (0, \delta)$ by showing that its determinant is of order $\mathcal{O}(\vartheta_2^{-4})$ as $s \to 0$, hence, $\det \mathcal{M}(s) \to \infty$ as $s \to 0$ (see [41, Lemma 5.5] for details), and therefore hypothesis (A6) holds for all $s \in (0, \delta)$, making $\delta > 0$ smaller if necessary. We can thus apply Theorem 4.1 for each $s \in (0, \delta)$, which gives the desired log-radially quasiperiodic solutions for (2.4), concluding the proof in the case $N \geq 3$.

The case $N = 2$ can be treated similarly. Instead of the Hamiltonian Φ as in (3.16), we start from the Hamiltonian (3.15). Since now $\mathcal{A} = 0$, the Hamiltonian is already diagonalized. The cubic terms are

$$\Phi_3(\xi', \eta') = \frac{1}{3} \sum_{j,k,\ell=1}^{2} \Theta(j, k, \ell) \xi'_j \xi'_k \xi'_\ell,$$

where

$$\Theta(j,k,\ell) = \frac{1}{3(\omega_j\omega_k\omega_\ell)^{1/2}} \int_{\mathbb{S}^{N-1}} a_2\varphi_j\varphi_k\varphi_\ell d\phi,$$

and there are no other cubic terms in Φ. One can reproduce the foregoing argument, with $\omega_1 > c_\delta > 0$ and $\omega_2 \to 0$ taking the role of ϑ_1 and ϑ_2, respectively, to derive the asymptotic behavior of the remaining terms of degree 2 (in I) and conclude that Φ_0 takes the form

$$\Phi_0(I;s) = \frac{C}{\omega_2^4(s)}\Big(\int_{\mathbb{S}^{N-1}} a_2(\phi;s)\varphi_2^3(\phi;s)d\phi\Big)^2 I_2^2 + \tilde{\Phi}(I;s) + \text{h.o.t.},$$

where C is a positive constant; the function $\tilde{\Phi}$, comprising all remaining quadratic terms in I, has coefficients of order $\mathcal{O}(\omega_2^{-7/2})$ as $s \to 0$; and h.o.t. stands for terms of degree 3 and higher in I. The rest of the argument is the same as in the previous case, the only change being that we use $\omega_2 \to 0$ rather than $\vartheta_2 \to 0$. This concludes the proof in the case $N = 2$, and the theorem has thus been proved. $\square$

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