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TRAVELING WAVE SOLUTIONS FOR AN EPIDEMIC MODEL

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Abstract. In this article, we consider a one-dimensional reaction-diffusion epidemic model, which is neither cooperative nor competitive. We study the possible impact of the spatial movement by investigating the existence of traveling wave solutions. We construct a pair of upper-lower solutions and then use Shauder's fixed point theorem to prove the existence of nonnegative nontrivial bounded semi-traveling wave solution. This is done by introducing a critical wave speed depending on the diffusion coefficients and other parameters in the model such that, for the wave speed that is greater than the critical wave speed, the model has such a solution. We also derive a condition under which the model has no nonnegative nontrivial bounded semi-traveling wave solution.

1. Introduction

Mathematical models play an important role in understanding the transmission and spread and estimating the impact of control measures of infectious disease like influenza, rabies, rubeola, malaria, dengue disease, COVID-19, etc. It turns out that the transmission and spread of almost all infectious diseases heavily depend on climate, rainfall, environment(temperature and humidity), social interactions, migration of population, and the spatial and genetic heterogeneity of host and parasite. In order to investigate the possible impact of the spatial movement of population, following [21], we consider one-dimensional reaction-diffusion epidemic model described by

$$\frac{\partial u_i}{\partial t} = d_i u_{ixx} + f_i(u), \quad 1 \le i \le n,\ x \in \mathbb{R},\ t > 0, \tag{1.1}$$

where u_i is the density of a population in compartment i and d_i is the diffusion coefficient of population i, f_i is the reaction term in compartment i under the influences of demographic process and epidemic interactions. $u = (u_1, u_2, \dots, u_n)^T$ with $u_i \ge 0$ represents the state of individuals in all compartments. Traveling waves are a common phenomenon in biology. The existence of traveling wave solutions for a reaction-diffusion system determines the long-term behavior of other solutions of the system and reveals a lot of information for the spread and transmission of the disease. In this paper, we introduce a critical wave speed $\tilde{c}$ such that (1.1)

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has a nonnegative nontrivial semi-traveling wave solution $u(t,x) = U(z)$, where $z = x + ct$, if $c \geq \tilde{c}$. In epidemiology, the existence and nonexistence of nontrivial traveling waves indicate whether an infectious disease could persist as a wave front of infection that travels geographically across vast distance [4].

Because of the general form of $f_i(u)$ in the model, (1.1) can be applied to a large class of compartmental epidemic models. It has been investigated from various aspects. The corresponding multi-dimensional version of model (1.1) on a bounded region with smooth boundary coupled with homogeneous Neumann boundary condition has been investigated in [21]. The corresponding reaction model has been investigated in [20].

One of the most important concerns about infectious disease is its ability to spread into a population. The minimal wave speed $c_{\min}$ for a traveling wave is a key parameter to characterize the speed at which the disease spreads in a spatial domain [4, 12, 24, 26]. Biologically speaking, epidemics can spread for $c \geq c_{\min}$ while they cannot spread for $c < c_{\min}$ [4]. Therefore, estimating the minimal wave speed $c_{\min}$ is a very significant work both theoretically and practically. It was first conjectured by Fisher in [8] that for model

$$u_t = du_{xx} + \rho u(1-u)$$

the minimal wave speed $c_{\min}$ equals the spreading speed c^* at which the region $\{x : u \sim 1\}$ takes over the set $\{x : u(x,0) = 0\}$ (see [17]). The conjecture was proved in [14]. Under certain assumptions, it has been proved (e.g. [2, 3]) that, for some more general models, the minimal wave speed $c_{\min}$ is equal to the spreading speed c^*. But usually, it is difficult to estimate c^*. When c^* is equal to the spreading speed $\bar{c}$ of the linearized system of (1.1) around the unique disease-free equilibrium, the spreading speed c^* of (1.1) is said to be linearly determined (see [15, 23]). If this is the case, we can estimate c^* as well as $c_{\min}$ by estimating $\bar{c}$. Usually, to estimate $\bar{c}$ is much easier than to estimate c^* or $c_{\min}$. For some models, $\bar{c}$ can be estimated quantitatively, see [15].

It is shown in [17, 23] that if (1.1) is cooperative, that is, $f_{iu_j} \geq 0$ for $j \neq i$, then under certain assumptions, (1.1) is linearly determined. This method has been applied to estimate the minimal wave speed $c_{\min}$ for a west nile virus model in [16] and a malaria model in [27].

If (1.1) is a competitive model, we can use an appropriate change of variables as done in [15] to change it to a cooperative system and then use the method in [17] to estimate the minimal wave speed $c_{\min}$.

Unfortunately, some systems are neither cooperative nor competitive like the models investigated in [4, 9, 10, 24, 25, 26]. For these models, the method proposed in [17] cannot be used to estimate the minimal wave speed $c_{\min}$.

In this article, we adapt the approach used in [4] (same method is also used in [10, 18, 19, 22, 26]) to introduce a critical wave speed $\tilde{c}$ and to derive some sufficient conditions under which semi-traveling wave solutions exist. The critical wave speed $\tilde{c}$ introduced here will play an important role in estimating the minimal wave speed $c_{\min}$. For some special cases, this critical wave speed coincides with the minimal wave speed of traveling wave solutions under appropriate assumptions. Because of the generality of our model, we have to overcome some technical difficulties by making appropriate changes.

To distinguish the disease-free states and infected states, following [20, 21], we let

$$u_I = (u_1, u_2, \dots, u_m)^T, \quad d_I = \operatorname{diag}(d_1, d_2, \dots, d_m),$$
$$u_S = (u_{m+1}, u_{m+2}, \dots, u_n)^T, \quad d_S = \operatorname{diag}(d_{m+1}, d_{m+2}, \dots, d_n),$$

where u_i, $1 \le i \le m$, represent the infected compartments, and u_i, $m+1 \le i \le n$ represent the uninfected compartments. Then we write (1.1) as

$$\begin{aligned} \frac{\partial u_I}{\partial t} &= d_I u_{Ixx} + f_I(u) \quad x \in \mathbb{R},\ t > 0, \\ \frac{\partial u_S}{\partial t} &= d_S u_{Sxx} + f_S(u), \quad x \in \mathbb{R},\ t > 0, \end{aligned} \tag{1.2}$$

where

$$f_I(u) = (f_1(u), f_2(u), \dots, f_m(u))^T, \quad f_S(u) = (f_{m+1}(u), f_{m+2}(u), \dots, f_n(u))^T.$$

We assume that $f_i(u)$ has the following form, for $i = 1, 2, \dots, m$,

$$\begin{aligned} f_i(u) &= f_i(u_1, u_2, \dots, u_n) \\ &= \Big(\sum_{j=1}^m a_{ij} u_j\Big)\Big(b_{i0} + \sum_{k=m+1}^n b_{ik} u_k\Big) - q_i u_i \\ &= A_i B_i - q_i u_i, \end{aligned} \tag{1.3}$$

and for $i = m+1, m+2, \dots, n$,

$$\begin{aligned} f_i(u) &= f_i(u_1, u_2, \dots, u_n) \\ &= Q_i - \Big(\sum_{j=1}^m a_{ij} u_j\Big)\Big(\sum_{k=m+1}^n b_{ik} u_k\Big) - q_i u_i \\ &= Q_i - A_i C_i - q_i u_i, \end{aligned} \tag{1.4}$$

where $Q_i (i = m+1, m+2, \dots, n)$ and $q_i (i = 1, 2, \dots, n)$ are positive constants, a_{ij}, b_{j0}, b_{ik} $(i = 1, 2, \dots, n,\ j = 1, 2, \dots m,\ k = m+1, m+2, \dots, n)$ are nonnegative constants and

$$A_i = A_i(u_I) = A_i(u_1, u_2, \dots, u_m) = \sum_{j=1}^m a_{ij} u_j, \quad i = 1, 2, \dots, n,$$

$$B_i = B_i(u_S) = B_i(u_{m+1}, u_{m+2}, \dots, u_n) = b_{i0} + \sum_{k=m+1}^n b_{ik} u_k, \quad i = 1, 2, \dots, m,$$

$$C_i = C_i(u_S) = C_i(u_{m+1}, u_{m+2}, \dots, u_n) = \sum_{k=m+1}^n b_{ik} u_k \quad i = m+1, m+2, \dots, n.$$

Such a model includes the diffusive influenza model with multiple strains

$$\begin{aligned} \frac{\partial I_1}{\partial t} &= d_1 I_{1xx} + (1-f)\beta_1(I_1 + \delta I_2)S - k_1 I_1, \quad x \in \mathbb{R},\ t > 0, \\ \frac{\partial I_2}{\partial t} &= d_2 I_{2xx} + f(1-r)\beta_1(I_1 + \delta I_2)S - k_2 I_2, \quad x \in \mathbb{R},\ t > 0, \\ \frac{\partial I_3}{\partial t} &= d_3 I_{3xx} + [fr\beta_1(I_1 + \delta I_2) + \beta_2 I_3]S - k_3 I_3, \quad x \in \mathbb{R},\ t > 0, \\ \frac{\partial S}{\partial t} &= d_S S_{xx} + \Lambda - [\beta_1(I_1 + \delta I_2) + \beta_2 I_3]S - \mu S, \quad x \in \mathbb{R},\ t > 0 \end{aligned} \tag{1.5}$$

which was investigated in [4]. The differential susceptibility epidemic model

$$\begin{aligned} \frac{\partial I}{\partial t} &= d_I I_{xx} + \eta\beta I \sum_{j=1}^{l} \alpha_j S_j - (\mu+\gamma) I \quad x \in \mathbb{R},\ t>0, \\ \frac{\partial S_i}{\partial t} &= d_i S_{ixx} + \mu p_i S^0 - \eta\beta\alpha_i I S_i - \mu S_i, \quad x \in \mathbb{R},\ t>0,\ i=1,2,\dots,l, \end{aligned} \tag{1.6}$$

whose corresponding reaction model was investigated in [13]. The within-host HIV model with cell-to-cell transmission,

$$\begin{aligned} \frac{\partial I_1}{\partial t} &= d_1 I_{1xx} + (\beta_1 I_1 + \beta_2 I_2) S - r I_1, \quad x \in \mathbb{R},\ t>0, \\ \frac{\partial I_2}{\partial t} &= d_2 I_{2xx} + \delta I_1 - \mu I_2, \quad x \in \mathbb{R},\ t>0, \\ \frac{\partial S}{\partial t} &= d_S S_{xx} + \lambda - (\beta_1 I_1 + \beta_2 I_2) S - \eta S, \quad x \in \mathbb{R},\ t>0 \end{aligned} \tag{1.7}$$

investigated in [19]. By direct computations, for $i, j = 1, 2, \dots, m$, we have

$$\begin{aligned} \frac{\partial f_i}{\partial u_j} &= a_{ij} B_i, \quad j \neq i, \\ \frac{\partial f_i}{\partial u_i} &= a_{ii} B_i - q_i\,; \end{aligned}$$

for $i = 1, 2, \dots, m$, $j = m+1, m+2, \dots, n$, we have

$$\frac{\partial f_i}{\partial u_j} = b_{ij} A_i\,;$$

for $i = m+1, m+2, \dots, n$, $j = 1, 2, \dots, m$, we have

$$\frac{\partial f_i}{\partial u_j} = -a_{ij} C_i\,;$$

and for $i, j = m+1, m+2, \dots, n$, we have

$$\begin{aligned} \frac{\partial f_i}{\partial u_j} &= -b_{ij} A_i, \quad j \neq i, \\ \frac{\partial f_i}{\partial u_i} &= -b_{ii} A_i - q_i. \end{aligned}$$

From these expressions we can see that model (1.2) is neither cooperative nor competitive.

This article is organized as follows. In next section we summarize some results for the corresponding reaction model that will be used in the following sections. In Section 3, we introduce a critical wave speed and establsih the existence of semi-traveling wave solutions. In Section 4, we prove a theorem of nonexistence of traveling wave solutions. Finally, in Section 5, we summarize what we did in this paper and suggest some possible directions of future work.

2. Basic reproduction number for the reaction model and some examples

When the spatial diffusion of population is omitted, the diffusive model (1.1) is reduced to the reaction model

$$\frac{du_i}{dt} = f_i(u), \quad i = 1, 2, \dots, n, \tag{2.1}$$

where $u_i = u_i(t)$ is independent of x. Since (1.1) is autonomous, the dynamics of (2.1) can give us some insight into the model (1.1). Therefore, we will cite some results about (2.1) from other references (see [20] and the references therein) and look at some specific examples. It is easily seen that (2.1) has a disease-free equilibrium $E_0 = (0, \dots, 0, u^0_{m+1}, \dots, u^0_n)^T$ with $u^0_i = \frac{Q_i}{q_i} > 0$, $m+1 \le i \le n$. The disease-free equilibrium is the population before the transmission of the disease. Write the linearized system of (2.1) around E_0 as

$$\frac{du}{dt} = Df(E_0)(u - E_0),$$

where $Df(E_0)$ is the derivative $[\frac{\partial f_i}{\partial u_j}]_{n\times n}$ evaluated at E_0. More specifically,

$$Df(E_0) = \begin{bmatrix} E - Q & 0 \\ -J & -M_S \end{bmatrix},$$

where

$$E = \begin{bmatrix} a_{11}B^0_1 & a_{12}B^0_1 & \cdots & a_{1m}B^0_1 \\ a_{21}B^0_2 & a_{22}B^0_2 & \cdots & a_{2m}B^0_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}B^0_m & a_{m2}B^0_m & \cdots & a_{mm}B^0_m \end{bmatrix}_{m\times m}$$

corresponds to the new infection,

$$Q = \begin{bmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \dots & q_m \end{bmatrix}_{m\times m}$$

corresponds to the remaining transfer terms,

$$J = \begin{bmatrix} a_{(m+1)1}C^0_{m+1} & a_{(m+1)2}C^0_{m+1} & \cdots & a_{(m+1)m}C^0_{m+1} \\ a_{(m+2)1}C^0_{m+2} & a_{(m+2)2}C^0_{m+2} & \cdots & a_{(m+2)m}C^0_{m+2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}C^0_n & a_{n2}C^0_n & \cdots & a_{nm}C^0_n \end{bmatrix}_{(n-m)\times m},$$

$$M_S = \begin{bmatrix} q_{m+1} & 0 & \cdots & 0 \\ 0 & q_{m+2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & q_n \end{bmatrix}_{(n-m)\times(n-m)},$$

where

$$\begin{gathered} B^0_i = B_i(u^0_S) = B_i(u^0_{m+1}, \dots, u^0_n), \quad i = 1, 2, \dots, m, \\ C^0_i = C_i(u^0_S) = C_i(u^0_{m+1}, \dots, u^0_n), \quad i = m+1, m+2, \dots, n. \end{gathered}$$

Recall that the basic reproduction number R_0 is the expected number of secondary infections generated by a single infectious individual during the infection period in an entirely susceptible population (see [1, 5, 6, 11, 20, 21]). To introduce R_0, following [20, 21] (also see the references therein), consider a "typical" infectious individual introduced into a completely susceptible population, to see how many new infections will be produced. Let $\phi_i(0)$ be the number of infected individuals initially in compartment i. Without reinfection, at any time t, let

$\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_m(t))^T$ be the total infective member which were developed from those initially infected individuals $\phi_i(0)$, then $\phi(t)$ satisfies

$$\phi'(t) = -Q\phi(t), \quad \phi(0) = (\phi_1(0), \phi_2(0), \dots, \phi_m(0))^T.$$

Therefore, $\phi(t) = e^{-Qt}\phi(0)$. Thus, the new infection at time t is $E\phi(t) = Ee^{-Qt}\phi(0)$. Consequently, the expected number of new infections produced by the initially infected individuals is

$$\int_0^\infty E\phi(t)dt = \int_0^\infty Ee^{-Qt}\phi(0)dt = EQ^{-1}\phi(0).$$

EQ^{-1} is called the next generation matrix for the model (see [6]). Following [6, 19, 20, 21], we introduce the basic reproduction number

$$R_0 = \rho(EQ^{-1}),$$

the spectral radius of EQ^{-1}. R_0 is the threshold value for the local stability of the disease-free equilibria of (2.1) as stated in the following theorem from [20] (see [5] for a similar result).

Theorem 2.1. *Let E_0 be a disease-free equilibrium of model* (2.1)*, then E_0 is locally asymptotically stable if $R_0 < 1$, but unstable if $R_0 > 1$.*

For model (1.5), $m = 3$, $n = 4$. For notation convenience, denote $\omega = \frac{\Lambda}{\mu}$, $\alpha = (1-f)\beta_1$, $\beta = f(1-r)\beta_1$, $\gamma = fr\beta_1$, then $E_0 = (0, 0, 0, \omega)$,

$$Df(E_0) = \begin{bmatrix} E-Q & 0 \\ -J & -M_S \end{bmatrix},$$

where

$$E = \omega \begin{bmatrix} \alpha & \alpha\delta & 0 \\ \beta & \beta\delta & 0 \\ \gamma & \gamma\delta & \beta_2 \end{bmatrix}, \quad Q = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix},$$

$$J = \begin{bmatrix} \beta_1\omega & \beta_1\delta\omega & \beta_2\omega \end{bmatrix}, \quad M_S = \mu.$$

It is easily seen that

$$EQ^{-1} = \omega \begin{bmatrix} \frac{\alpha}{k_1} & \frac{\alpha\delta}{k_2} & 0 \\ \frac{\beta}{k_1} & \frac{\beta\delta}{k_2} & 0 \\ \frac{\gamma}{k_1} & \frac{\gamma\delta}{k_2} & \frac{\beta_2}{k_3} \end{bmatrix}$$

and its two nonzero eigenvalues are

$$\lambda_1 = \omega\Big(\frac{\alpha}{k_1} + \frac{\beta\delta}{k_2}\Big),$$

which is denoted by R_{SC} in [4], representing the number of secondary sensitive cases that one individual infected with the sensitive strain initiates in a completely susceptible population where antiviral treatment is implemented, and

$$\lambda_2 = \frac{\beta_2\omega}{k_3},$$

which is denoted by R_{RC} in [4], representing the number of secondary resistant cases that one individual infected with the resistant strain initiates in a completely susceptible population.

$$\rho(EQ^{-1}) = \max\{\lambda_1, \lambda_2\}$$

is denoted by R_C in [4] and is called the control reproduction number of the corresponding reaction model, which is used to determine whether the epidemic can be contained when certain control measures are taken (also see [1]).

For model (1.6), $m = 1$, $n = l + 1$. Denote $\omega = \eta\beta S^0$, $q = \mu + \gamma$, then $E_0 = (0, p_1 S^0, p_2 S^0, \dots, p_l S^0)$,

$$Df(E_0) = \begin{bmatrix} \omega\sum_{j=1}^{l}\alpha_j p_j - q & 0 & 0 & 0 & \cdots & 0 \\ -\omega\alpha_1 p_1 & -\mu & 0 & 0 & \cdots & 0 \\ -\omega\alpha_2 p_2 & 0 & -\mu & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\omega\alpha_l p_l & 0 & 0 & 0 & \cdots & -\mu \end{bmatrix}.$$

$$\rho(EQ^{-1}) = EQ^{-1} = \frac{\omega\sum_{j=1}^{l}\alpha_j p_j}{q} = \frac{\eta\beta S^0\sum_{j=1}^{l}\alpha_j p_j}{\mu+\gamma},$$

which is the reproductive number R_0 introduced in [13].

For model (1.7), $m = 2$, $n = 3$, we denote $\omega = \frac{\lambda}{\mu}$. Then $E_0 = (0, 0, \omega)$ and

$$Df(E_0) = \begin{bmatrix} E - Q & 0 \\ -J & -M_S \end{bmatrix},$$

where

$$E = \begin{bmatrix} \beta_1\omega & \beta_2\omega \\ \delta & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} r & 0 \\ 0 & \mu \end{bmatrix}, \quad J = [\beta_1\omega \ \ \beta_2\omega], \quad M_S = \eta.$$

It is easily seen that

$$EQ^{-1} = \begin{bmatrix} \frac{\beta_1\omega}{r} & \frac{\beta_2\omega}{\mu} \\ \frac{\delta}{r} & 0 \end{bmatrix}$$

and its two eigenvalues are

$$\lambda = \frac{\beta_1\lambda \pm \sqrt{\beta_1^2\lambda^2 + 4\beta_2\delta r\lambda}}{2r\mu}.$$

Thus,

$$\rho(EQ^{-1}) = \frac{\beta_1\lambda + \sqrt{\beta_1^2\lambda^2 + 4\beta_2\delta r\lambda}}{2r\mu}.$$

3. Semi-traveling waves for the diffusive model

The existence of traveling wave solutions for reaction-diffusion models is one of the most important topics for the past several decades. To investigate the existence of traveling wave solutions for (1.1), we start with the existence of so-called semi-traveling waves as defined in the following definition from [4, 12, 26].

Definition 3.1. A solution $u(t, x)$ of (1.2) of the form

$$u(t, x) = U(z),$$

where $z = x + ct$, is called a semi-traveling wave solution connected to the disease-free equilibrium E_0 if it satisfies

$$\lim_{z\to-\infty} U(z) = E_0. \tag{3.1}$$

From the definition we can see that a semi-traveling wave solution is a traveling wave solution starting from the disease-free equilibrium. Such a solution is of biological significance since we can get a lot of information from it, such as whether epidemics will spread, asymptotic speed of propagation, and the final state of the wavefront, etc.

To prove the existence of semi-traveling wave solution for (1.1), we adapt the approach used in [4, 10, 18, 19, 22, 24, 25, 26]. That is, first we introduce an auxiliary system, then construct a pair of upper-lower solutions for the auxiliary system by linearizing the corresponding wave equation at the disease-free equilibrium. Then, use the Schauder's fixed point theorem to prove the existence of semi-traveling wave solution for the auxiliary system. Finally, by taking the limit, we obtain the existence of semi-traveling wave solution for (1.1).

We consider the following auxiliary system related to (1.2),

$$\begin{gathered}\frac{\partial u_I}{\partial t} = d_I u_{Ixx}, + f_I(u) - \delta I_m u_I^2 \quad x \in \mathbb{R},\ t > 0, \\ \frac{\partial u_s}{\partial t} = d_S u_{Sxx} + f_S(u), \quad x \in \mathbb{R},\ t > 0,\end{gathered} \tag{3.2}$$

where $u_I^2 = (u_1^2, u_2^2, \dots, u_m^2)^T$, $\delta > 0$ is a small number, and I_m is the identity matrix of size m. Let $u(t,x) = U(z)$ be a semi-traveling wave solution of (3.2), then U satisfies

$$\begin{gathered}cU_I' = d_I U_I'' + f_I(U) - \delta I_m U_I^2, \\ cU_S' = d_S U_S'' + f_S(U),\end{gathered} \tag{3.3}$$

where $U_I = (U_1, U_2, \dots, U_m)^T$ and $U_S = (U_{m+1}, U_{m+2}, \dots, U_n)^T$. The limiting equations of (3.3) as $\delta \to 0$ are the wave equations corresponding to (1.2), especially, the equation involving U_I only is the equation

$$cU_I' = d_I U_I'' + f_I(U) \tag{3.4}$$

Linearizing (3.4) around the disease-free equilibrium E_0 leads to the linear system

$$c\Phi_I' = d_I \Phi_I'' + E\Phi_I - Q\Phi_I, \tag{3.5}$$

where $\Phi_I = (\phi_1, \phi_2, \dots, \phi_m)^T$. Next we are going to introduce a critical wave speed $\tilde{c}$. To do this, we adopt the method used in [4]. Let $(\phi_1(z), \phi_2(z), \dots, \phi_m(z)) = e^{\lambda z}(v_1, v_2, \dots, v_m)$, then system (3.5) is equivalent to

$$\mathcal{A}(v_1, v_2, \dots, v_m)^T = c\lambda(v_1, v_2, \dots, v_m)^T, \tag{3.6}$$

where

$$\mathcal{A} = \lambda^2 d_I + E - Q.$$

Denoting $\mathbf{V} = (v_1, v_2, \dots, v_m)^T$, we can write (3.6) as

$$(\lambda^2 d_I - c\lambda I_m - Q)\mathbf{V} = -E\mathbf{V}$$

or

$$(-\lambda^2 Q^{-1} d_I + c\lambda Q^{-1} + I_m)\mathbf{V} = Q^{-1}E\mathbf{V}.$$

By direct computations, we have

$$\begin{aligned}\mathcal{B}(\lambda, c) &= -\lambda^2 Q^{-1} d_I + c\lambda Q^{-1} + I_m \\ &= \operatorname{diag}\Big(\frac{-d_1\lambda^2 + c\lambda + q_1}{q_1}, \dots, \frac{-d_m\lambda^2 + c\lambda + q_m}{q_m}\Big).\end{aligned}$$

Assume $\lambda > 0$ such that $-d_i\lambda^2 + c\lambda + q_i > 0$. Then $\mathcal{B}(\lambda, c)$ is invertible and the above equation can be written as

$$B(\lambda, c)^{-1}(Q^{-1}E)\mathbf{V} = \mathbf{V}.$$

Denoting $\mathcal{H}(\lambda, c) = B(\lambda, c)^{-1}(Q^{-1}E)$, we have

$$\mathcal{H}(\lambda, c)\mathbf{V} = \mathbf{V}. \tag{3.7}$$

Let $d = \max_{1\le i\le m} d_i > 0$. Observe that, for $c > 0$, the two roots of $-d_i\lambda^2+c\lambda+q_i = 0$ are

$$\lambda_1 = \frac{c - \sqrt{c^2 + 4d_iq_i}}{2d_i} < 0,$$

and

$$\lambda_2 = \frac{c + \sqrt{c^2 + 4d_iq_i}}{2d_i} > \frac{c}{d_i} \ge \frac{c}{d} > 0,$$

we know that, for $\lambda \in [0,\ \frac{c}{d}]$, $\mathcal{B}(\lambda, c)$ is invertible and

$$\mathcal{B}(\lambda, c)^{-1} = \operatorname{diag}\Big(\frac{q_1}{-d_1\lambda^2 + c\lambda + q_1}, \dots, \frac{q_m}{-d_m\lambda^2 + c\lambda + q_m}\Big),$$

and $\mathcal{H}(\lambda, c)$ is a nonnegative matrix.

Let $p_{ci}(\lambda) = -d_i\lambda^2 + c\lambda + q_i$, then

$$\gamma_i(c) = p_{ci}(\frac{c}{2d}) = \frac{(2d - d_i)c^2}{4d^2} + q_i > 0$$

and which is strictly increasing for $c \in [0, \infty)$. Thus, $\mathcal{H}(\frac{c}{2d}, c)$ is strictly decreasing for $c \in [0, \infty)$ and $\mathcal{H}(0, 0) = Q^{-1}E$.

Denote by $\rho(\mathcal{H}(\lambda, c))$ the principal eigenvalue of the nonnegative matrix $\mathcal{H}(\lambda, c)$ for $\lambda \in [0, \frac{c}{2d}]$. Since $\rho(\mathcal{H}(\lambda, c))$ is continuous and monotonically increasing with respect to the nonnegative matrix $\mathcal{H}(\lambda, c)$, $\rho(\mathcal{H}(\frac{c}{2d}, c))$ is strictly decreasing for $c \in [0, \infty)$ with $\rho(\mathcal{H}(0, 0)) = \rho(Q^{-1}E)$ and

$$\lim_{c\to\infty} \rho\Big(\mathcal{H}(\frac{c}{2d}, c)\Big) = 0.$$

It can be shown that $\rho(Q^{-1}E) = \rho(EQ^{-1})$(see [4]). Therefore, if $R_0 = \rho(EQ^{-1}) = \rho(Q^{-1}E) = \rho(\mathcal{H}(0, 0)) > 1$, by the continuity and monotonicity of $\rho(\mathcal{H}(\frac{c}{2d}, c))$ with respect to c, there exists a unique $\tilde{c} > 0$ such that $\rho(\mathcal{H}(\frac{\tilde{c}}{2d}, \tilde{c})) = 1$ and for $c \in [0, \tilde{c})$, $\rho(\mathcal{H}(\frac{c}{2d}, c)) > 1$. For $c \in (\tilde{c}, \infty)$, $\rho(\mathcal{H}(\frac{c}{2d}, c)) < 1$.

For any $c > \tilde{c}$ fixed, since $p'_{ci}(\lambda) = c - 2d_i\lambda > 0$ for $\lambda \in \big[0, \frac{c}{2d}\big]$, $\rho(\mathcal{H}(\lambda, c))$ is strictly decreasing and nonnegative for $\lambda \in \big[0, \frac{c}{2d}\big]$. But,

$$\rho(\mathcal{H}(0, c)) = \rho(\mathcal{H}(0, 0)) = R_0 > 1$$

and $\rho(\mathcal{H}(\frac{c}{2d}, c)) < 1$, by the continuity and monotonicity of $\rho(\mathcal{H}(\lambda, c))$ with respect to λ, there exists a unique $\lambda_c \in \big(0, \frac{c}{2d}\big)$ such that $\rho(\mathcal{H}(\lambda_c, c)) = 1$ and for $\lambda \in [0, \lambda_c)$, $\rho(\mathcal{H}(\lambda, c)) > 1$, and for $\lambda \in \big(\lambda_c, \frac{c}{2d}\big]$, $\rho(\mathcal{H}(\lambda, c)) < 1$. Thus we have the following lemma (see [4, Lemmas A.1 and A.2]).

Lemma 3.2. *If $R_0 = \rho(EQ^{-1}) > 1$, then, there exists a unique $\tilde{c} > 0$ such that for any $c > \tilde{c}$, there exists a unique $\lambda_c \in \big(0, \frac{c}{2d}\big)$ and $\mathbf{V}_c = (v_1, v_2, \dots, v_m)^T$ with $v_i > 0$, $1 \le i \le m$, such that*

$$\mathcal{H}(\lambda_c, c)\mathbf{V}_c = \mathbf{V}_c.$$

The vector $\Phi_I = (\phi_1(z), \phi_2(z), \dots, \phi_m(z))^T$ *with* $\phi_i(z) = v_i e^{\lambda_c z}$, $i = 1, 2, \dots, m$, *satisfies* (3.5).

Now we adapt the approach in [4] with necessary changes to construct a pair of upper-lower solutions of (3.3). That is, we define

$$\overline{U_S}(z) = (u^0_{m+1}, \dots, u^0_n)^T,$$
$$\underline{U_S}(z) = (\underline{u_{m+1}}(z), \dots, \underline{u_n}(z))^T,$$

with $\underline{u_i}(z) = \max\{u^0_i(1 - \eta e^{\alpha z}), 0\}$ for $m+1 \le i \le n$, and

$$\overline{U_I}(z) = (\overline{u_1}(z), \dots, \overline{u_m}(z))^T,$$

with $\overline{u_i}(z) = \min\{v_i e^{\lambda_c z}, v_i \omega\}$ for $1 \le i \le m$,

$$\underline{U_I}(z) = (\underline{u_1}(z), \dots, \underline{u_m}(z))^T,$$

with $\underline{u_i}(z) = \max\{v_i e^{\lambda_c z}(1 - \tau e^{\mu z}), 0\}$ for $1 \le i \le m$, where constants $\eta, \alpha, \omega, \tau, \mu$ are to be determined later.

Lemma 3.3. *For* $\omega > 1$ *large enough,* $\overline{U_I}(z)$ *satisfies*

$$c\overline{U_I}' \ge d_I \overline{U_I}'' + f_I(\overline{U}) - \delta I_m \overline{U_I}^2, \tag{3.8}$$

where $\overline{U} = \overline{U}(z) = (\overline{U_I}(z), \overline{U_S}(z))^T$.

Proof. From the definition of $\overline{u_i}(z) = \min\{v_i e^{\lambda_c z}, v_i \omega\}$ for $1 \le i \le m$, if z such that $e^{\lambda_c z} \le \omega$, i.e. $z \le \frac{\ln \omega}{\lambda_c} \triangleq z_1$, then $\overline{u_i}(z) = v_i e^{\lambda_c z}$ and $\overline{U_I}(z) = \Phi_I$. Then, from the definition of E, Q, and $\overline{U}$, we have $f_I(\overline{U}) = (E - Q)\Phi_I$. Therefore, from Lemma 3.2 and (3.5), we have

$$d_I \overline{U_I}'' - c\overline{U_I}' + f_I(\overline{U}) = d_I \Phi_I'' - c\Phi_I' + E\Phi_I - Q\Phi_I = 0 \le \delta I_m \overline{U_I}^2.$$

That is, (3.8) holds.

If $z > z_1$, then $\overline{u_i}(z) = v_i \omega$ and $\overline{U_I}(z) = \omega \mathbf{V}_c$. Then

$$d_I \overline{U_I}'' - c\overline{U_I}' + f_I(\overline{U}) - \delta I_m \overline{U_I}^2 = f_I(\overline{U}) - \delta I_m (v_1^2, \dots, v_m^2)^T \omega^2.$$

But, for $i = 1, 2, \dots, m$,

$$\begin{aligned} f_i(\overline{U}) - \delta v_i^2 \omega^2 &= \sum_{j=1}^m a_{ij} v_j \omega B^0_i - q_i v_i \omega - \delta v_i^2 \omega^2 \\ &= (\sum_{j=1}^m a_{ij} v_j B^0_i - q_i v_i - \delta v_i^2 \omega)\omega. \end{aligned}$$

Therefore, if we take $\omega > 1$ such that

$$\omega > \max_{1 \le i \le m} \frac{\sum_{j=1}^m a_{ij} v_j B^0_i - q_i v_i}{\delta v_i^2},$$

then

$$d_I \overline{U_I}'' - c\overline{U_I}' + f_I(\overline{U}) - \delta I_m \overline{U_I}^2 \le 0$$

and (3.8) holds. This completes the proof. □

Lemma 3.4. *For* $0 < \alpha < \min\{\frac{c}{D}, \lambda_c\}$ *and*
$$\eta > \max\Big\{1, \max_{m+1\le i\le n} \frac{(\sum_{j=1}^m a_{ij}v_j)(\sum_{k=m+1}^n b_{ik}u_k^0)}{q_iu_i^0}\Big\},$$
$\underline{U_S}(z)$ *satisfies*
$$c\underline{U_S}' \le d_S\underline{U_S}'' + f_S(\tilde{U}), \tag{3.9}$$
where $D = \max\{d_{m+1}, \dots, d_n\}$ *and* $\tilde{U} = \tilde{U}(z) = (\overline{U_I}(z), \underline{U_S}(z))^T$.

Proof. Let $z_2 = -\frac{\ln\eta}{\alpha} < 0$. If z satisfies $z \ge z_2$, then $\underline{u_i}(z) = 0$ for $m+1 \le i \le n$, $\underline{U_S}(z) = (0, \dots, 0)^T$ and $\tilde{U} = (\overline{U_I}(z), 0, \dots, 0)^T$. Thus, $f_S(\tilde{U}) > 0$. Therefore, (3.9) holds.

If $z \le z_2$, $\underline{u_i}(z) = u_i^0(1-\eta e^{\alpha z})$ for $m+1 \le i \le n$. Observe that $z_2 < 0 < z_1$, $\overline{u_i}(z) = v_ie^{\lambda_c z}$ for $1 \le i \le m$. Thus,
$$\tilde{U} = (v_1e^{\lambda_c z}, \dots, v_me^{\lambda_c z}, u_{m+1}^0(1-\eta e^{\alpha z}), \dots, u_n^0(1-\eta e^{\alpha z}))^T,$$
and
$$d_S\underline{U_S}'' - c\underline{U_S}' + f_S(\tilde{U}) = (-d_S\alpha^2 + c\alpha)\eta e^{\alpha z}\Theta + f_S(\tilde{U}),$$
where $\Theta = (u_{m+1}^0, \dots, u_n^0)^T$.

Since, for $m+1 \le i \le n$,
$$\begin{aligned}
&f_i(\tilde{U})\\
&= Q_i - \Big(\sum_{j=1}^m a_{ij}v_je^{\lambda_c z}\Big)\Big(\sum_{k=m+1}^n b_{ik}u_k^0(1-\eta e^{\alpha z})\Big) - q_iu_i^0(1-\eta e^{\alpha z})\\
&= Q_i - e^{\lambda_c z}(1-\eta e^{\alpha z})\Big(\sum_{j=1}^m a_{ij}v_j\Big)\Big(\sum_{k=m+1}^n b_{ik}u_k^0\Big) - q_iu_i^0 + q_i\eta u_i^0e^{\alpha z}\\
&= \eta e^{\lambda_c z}e^{\alpha z}\Big(\sum_{j=1}^m a_{ij}v_j\Big)\Big(\sum_{k=m+1}^n b_{ik}u_k^0\Big) - e^{\lambda_c z}\Big(\sum_{j=1}^m a_{ij}v_j\Big)\Big(\sum_{k=m+1}^n b_{ik}u_k^0\Big) + q_i\eta u_i^0e^{\alpha z}\\
&\ge -e^{\lambda_c z}\Big(\sum_{j=1}^m a_{ij}v_j\Big)\Big(\sum_{k=m+1}^n b_{ik}u_k^0\Big) + q_i\eta u_i^0e^{\alpha z},
\end{aligned}$$
we have
$$\begin{aligned}
&d_iu_i'' - cu_i' + f_i(\tilde{U})\\
&\ge (-d_i\alpha^2\eta u_i^0 + c\alpha\eta u_i^0)e^{\alpha z} - e^{\lambda_c z}\Big(\sum_{j=1}^m a_{ij}v_j\Big)\Big(\sum_{k=m+1}^n b_{ik}u_k^0\Big) + q_i\eta u_i^0e^{\alpha z}\\
&= \Big[-d_i\alpha^2\eta u_i^0 + c\alpha\eta u_i^0 + q_i\eta u_i^0 - e^{(\lambda_c-\alpha)z}\Big(\sum_{j=1}^m a_{ij}v_j\Big)\Big(\sum_{k=m+1}^n b_{ik}u_k^0\Big)\Big]e^{\alpha z}\\
&\ge \Big[(-d_i\alpha^2 + c\alpha + q_i)\eta u_i^0 - \Big(\sum_{j=1}^m a_{ij}v_j\Big)\Big(\sum_{k=m+1}^n b_{ik}u_k^0\Big)\Big]e^{\alpha z} \ge 0
\end{aligned}$$
for $0 < \alpha < \min\{\frac{c}{D}, \lambda_c\}$ and
$$\eta > \frac{(\sum_{j=1}^m a_{ij}v_j)(\sum_{k=m+1}^n b_{ik}u_k^0)}{q_iu_i^0}.$$
Therefore, (3.9) holds. This completes the proof. $\square$

Lemma 3.5. *Let $0 < \mu < \alpha < \lambda_c$ small enough such that $\lambda_c + \mu < \frac{c}{2d}$. Then for $\tau > 1$ large enough, $\underline{U_I}(z)$ satisfies*
$$c\underline{U_I}' \le d_I \underline{U_I}'' + f_I(\underline{U}) - \delta I_m \underline{U_I}^2, \tag{3.10}$$
where $\underline{U} = \underline{U}(z) = (\underline{U_I}(z), \underline{U_S}(z))^T$.

Proof. If $1 - \tau e^{\mu z} \le 0$, that is, $z \ge -\frac{\ln \tau}{\mu} \triangleq z_3$, $\underline{U_I} = (0, \dots, 0)$. Then, we have $f_I(\underline{U}) = 0$ and (3.10) holds. For $z < z_3$, $\underline{u_i}(z) = v_i e^{\lambda_c z}(1 - \tau e^{\mu z})$ for $1 \le i \le m$. For $\tau > 1$ large enough (depending on α and η), we have $z_3 < z_2 < 0$. Thus, for $z < z_3$, $\underline{u_i}(z) = u_i^0(1 - \eta e^{\alpha z})$ for $m + 1 \le i \le n$ and $\underline{U} = (v_1 e^{\lambda_c z}(1 - \tau e^{\mu z}), \dots, v_m e^{\lambda_c z}(1 - \tau e^{\mu z}), u_{m+1}^0(1 - \eta e^{\alpha z}), \dots, u_n^0(1 - \eta e^{\alpha z}))^T$. Thus, for $1 \le i \le m$,
$$\underline{u_i}'(z) = v_i[\lambda_c(1 - \tau e^{\mu z}) - \tau\mu e^{\mu z}]e^{\lambda_c z},$$
and
$$\underline{u_i}''(z) = v_i[\lambda_c^2(1 - \tau e^{\mu z}) - (2\lambda_c\tau\mu + \tau\mu^2)e^{\mu z}]e^{\lambda_c z}.$$
Therefore, for $1 \le i \le m$, we have
$$\begin{aligned}
&cu_i' - d_i u_i'' - f_i(\underline{U}) + \delta u_i^2 \\
&= cv_i[\lambda_c(1 - \tau e^{\mu z}) - \tau\mu e^{\mu z}]e^{\lambda_c z} \\
&\quad - d_i v_i[\lambda_c^2(1 - \tau e^{\mu z}) - (2\lambda_c\tau\mu + \tau\mu^2)e^{\mu z}]e^{\lambda_c z} \\
&\quad - \Big[\sum_{j=1}^m a_{ij} v_j e^{\lambda_c z}(1 - \tau e^{\mu z})\Big(b_{i0} + \sum_{k=m+1}^n b_{ik} u_k^0(1 - \eta e^{\alpha z})\Big)\Big] \\
&\quad + q_i v_i e^{\lambda_c z}(1 - \tau e^{\mu z}) + \delta v_i^2 e^{2\lambda_c z}(1 - \tau e^{\mu z})^2 \\
&= e^{\lambda_c z}(1 - \tau e^{\mu z})\Big[- d_i v_i \lambda_c^2 + cv_i\lambda_c - \Big(\sum_{j=1}^m a_{ij} v_j\Big)\Big(b_{i0} + \sum_{k=m+1}^n b_{ik} u_k^0\Big) + q_i v_i\Big] \\
&\quad + [d_i v_i(2\lambda_c\tau\mu + \tau\mu^2) - cv_i\tau\mu]e^{(\lambda_c + \mu)z} \\
&\quad + \eta e^{\lambda_c z}(1 - \tau e^{\mu z})e^{\alpha z}\Big(\sum_{j=1}^m a_{ij} v_j\Big)\Big(\sum_{k=m+1}^n b_{ik} u_k^0\Big) + \delta v_i^2 e^{2\lambda_c z}(1 - \tau e^{\mu z})^2.
\end{aligned}$$
Since $\Phi_I = (v_1 e^{\lambda_c z}, \dots, v_m e^{\lambda_c z})^T$ satisfies (3.5), we have
$$-d_i v_i \lambda_c^2 e^{\lambda_c z} + cv_i\lambda_c e^{\lambda_c z} - \Big(\sum_{j=1}^m a_{ij} v_j e^{\lambda_c z}\Big)\Big(b_{i0} + \sum_{k=m+1}^n b_{ik} u_k^0\Big) + q_i v_i e^{\lambda_c z} = 0,$$
or
$$-d_i v_i \lambda_c^2 + cv_i\lambda_c - \Big(\sum_{j=1}^m a_{ij} v_j\Big)\Big(b_{i0} + \sum_{k=m+1}^n b_{ik} u_k^0\Big) + q_i v_i = 0.$$
Therefore,
$$\begin{aligned}
&cu_i' - d_i u_i'' - f_i(\underline{U}) + \delta u_i^2 \\
&= [d_i v_i(2\lambda_c\tau\mu + \tau\mu^2) - cv_i\tau\mu]e^{(\lambda_c + \mu)z} \\
&\quad + \eta e^{\lambda_c z}(1 - \tau e^{\mu z})e^{\alpha z}\Big(\sum_{j=1}^m a_{ij} v_j\Big)\Big(\sum_{k=m+1}^n b_{ik} u_k^0\Big) + \delta v_i^2 e^{2\lambda_c z}(1 - \tau e^{\mu z})^2 \\
&= e^{(\lambda_c + \mu)z}\Big\{[d_i(2\lambda_c + \mu) - c]v_i\tau\mu
\end{aligned}$$

$$+\eta(1-\tau e^{\mu z})e^{(\alpha-\mu)z}\Big(\sum_{j=1}^{m}a_{ij}v_j\Big)\Big(\sum_{k=m+1}^{n}b_{ik}u_k^0\Big)+\delta v_i^2 e^{(\lambda_c-\mu)z}(1-\tau e^{\mu z})^2\Big\}.$$

Observe that, for $z < z_3 = -\frac{\ln\tau}{\mu}$, we have $0 < 1-\tau e^{\mu z} < 1$ and $e^z < e^{-\frac{\ln\tau}{\mu}}$. Thus, for $0<\mu<\alpha<\lambda_c$,

$$e^{(\alpha-\mu)z} < e^{-(\alpha-\mu)\frac{\ln\tau}{\mu}},$$

and

$$e^{(\lambda_c-\mu)z} < e^{-(\lambda_c-\mu)\frac{\ln\tau}{\mu}}.$$

Hence,

$$\begin{aligned}
&cu_i' - d_iu_i'' - f_i(\underline{U}) + \delta u_i^2\\
&\le e^{(\lambda_c+\mu)z}\Big\{[d_i(2\lambda_c+\mu)-c]v_i\tau\mu\\
&\quad + \eta e^{-(\alpha-\mu)\frac{\ln\tau}{\mu}}\Big(\sum_{j=1}^{m}a_{ij}v_j\Big)\Big(\sum_{k=m+1}^{n}b_{ik}u_k^0\Big)+\delta v_i^2 e^{-(\lambda_c-\mu)\frac{\ln\tau}{\mu}}\Big\}.
\end{aligned}$$

Since $\lambda_c < \frac{c}{2d}$, we can choose $\mu>0$ small enough such that

$$\lambda_c+\mu < \frac{c}{2d}.$$

Then

$$d_i(2\lambda_c+\mu) < d(2\lambda_c+\mu) < 2d(\lambda_c+\mu) < c.$$

Therefore, $d_i(2\lambda_c+\mu)-c<0$, and we can choose $\tau>1$ large enough so that

$$\begin{aligned}
&[d_i(2\lambda_c+\mu)-c]v_i\tau\mu\\
&+\eta e^{-(\alpha-\mu)\frac{\ln\tau}{\mu}}\Big(\sum_{j=1}^{m}a_{ij}v_j\Big)\Big(\sum_{k=m+1}^{n}b_{ik}u_k^0\Big)+\delta v_i^2 e^{-(\lambda_c-\mu)\frac{\ln\tau}{\mu}} \le 0.
\end{aligned}$$

Thus, $cu_i' - d_iu_i'' - f_i(\underline{U}) + \delta u_i^2 \le 0$, and (3.10) holds. This completes the proof. □

Now we use Schauder's fixed point theorem to prove the existence of semi-traveling wave solutions of (3.2). To do this, we introduce Banach space $C_\sigma(\mathbb{R},\mathbb{R}^n)$ such that for $U=(u_1(z),\dots,u_n(z))^T\in C_\sigma(\mathbb{R},\mathbb{R}^n)$, its norm is defined by

$$\|U\| \triangleq |U(\cdot)|_\sigma = \max_{1\le i\le n}\sup_{z\in\mathbb{R}}|u_i(z)|e^{-\sigma|z|},$$

where $\sigma>0$ will be determined later. We are going to look for semi-traveling wave solutions $U(z)=(U_I(z),U_S(z))^T\in C_\sigma(\mathbb{R},\mathbb{R}^n)$ of (3.2) satisfying

$$\underline{U_I}(z)\le U_I(z)\le \overline{U_I}(z),\quad \underline{U_S}(z)\le U_S(z)\le \overline{U_S}(z).$$

Consider the closed and convex subset Σ of $C_\sigma(\mathbb{R},\mathbb{R}^n)$ given by

$$\Sigma=\{U=(U_I,U_S)^T\in C_\sigma(\mathbb{R},\mathbb{R}^n),\ \underline{U_I}\le U_I\le\overline{U_I},\ \underline{U_S}\le U_S\le\overline{U_S}\},$$

and define map $\Gamma=(\Gamma_I,\Gamma_S):\Sigma\to C_\sigma(\mathbb{R},\mathbb{R}^n)$ as follows:

For $U(z)=(U_I(z),U_S(z))^T\in\Sigma$:

$$\begin{gathered}
\Gamma(U)=(\Gamma_I(U),\Gamma_S(U))=(\Gamma_1(U),\dots,\Gamma_m(U),\Gamma_{m+1}(U),\dots,\Gamma_n(U)),\\
\Gamma_I(U)=F_IU_I+f_I(U)-\delta I_mU_I^2,\\
\Gamma_S(U)=F_SU_S+f_S(U),
\end{gathered}$$

where

$$F_I = \begin{bmatrix} \xi_1 & 0 & 0 & \dots & 0 \\ 0 & \xi_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & \xi_m \end{bmatrix}_{m\times m},$$

$$F_S = \begin{bmatrix} \xi_{m+1} & 0 & 0 & \dots & 0 \\ 0 & \xi_{m+2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & \xi_n \end{bmatrix}_{(n-m)\times(n-m)},$$

with $\xi_i > q_i$ large enough such that for $U \in \Sigma$, $\Gamma_i(U) \geq 0$, which will be determined later. Then system (3.3) can be written as

$$-d_i u_i''(z) + cu_i'(z) + \xi_i u_i(z) = \Gamma_i(U)(z), \tag{3.11}$$

for $i = 1, 2, \dots n$.

Let $x_{i1} < 0 < x_{i2}$ be the two distinct roots of $d_i x_i^2 - cx_i - \xi_i = 0$, then $x_i = x_{i2} - x_{i1} > 0$. We define map $\Pi = (\Pi_1, \dots, \Pi_n)^T : \Sigma \to C_\sigma(\mathbb{R}, \mathbb{R}^n)$ with $\Pi_i : \Sigma \to C_\sigma(\mathbb{R}, \mathbb{R})$ given by

$$\Pi_i(U)(z) = \frac{1}{d_i x_i}\Big[\int_{-\infty}^{z} e^{x_{i1}(z-s)}\Gamma_i(U)(s)ds + \int_{z}^{\infty} e^{x_{i2}(z-s)}\Gamma_i(U)(s)ds\Big],$$

then it holds

$$-d_i\Pi_i''(z) + c\Pi_i'(z) + \xi_i\Pi_i(z) = \Gamma_i(U)(z). \tag{3.12}$$

It is easy to see that any fixed point of Π is a solution of (3.11), which is a semi-traveling wave solution of (3.2). On the other hand, a solution of (3.11) is a fixed point of operator Π .

Lemma 3.6. *The operator Π maps Σ into Σ and it is continuous and compact with respect to the norm $|\cdot|_\sigma$ in $B_\sigma(\mathbb{R}, \mathbb{R}^n)$, where*

$$B_\sigma(\mathbb{R}, \mathbb{R}^n) = \{U(z) \in C_\sigma(\mathbb{R}, \mathbb{R}^n) : |U(\cdot)|_\sigma < \infty\}.$$

Proof. We first prove that Π maps Σ into Σ. Indeed, if $U(z) = (U_I(z), U_S(z))^T \in \Sigma$, that is,

$$\underline{U_I}(z) \leq U_I(z) \leq \overline{U_I}(z), \quad \underline{U_S}(z) \leq U_S(z) \leq \overline{U_S}(z),$$

we need to prove that

$$\underline{u_i}(z) \leq \Pi_i(z) \leq \overline{u_i}(z), \quad i = 1, 2, \dots, n. \tag{3.13}$$

Recall that, for $i = 1, 2, \dots, m$,

$$\overline{u_i}(z) = \begin{cases} v_i e^{\lambda_c z}, & z \leq z_1 = \frac{\ln\omega}{\lambda_c}, \\ v_i\omega, & z > z_1, \end{cases}$$

and

$$\underline{u_i}(z) = \begin{cases} v_i e^{\lambda_c z}(1 - \tau e^{\mu z}), & z \leq z_3 = -\frac{\ln\tau}{\mu}, \\ 0, & z > z_3, \end{cases}$$

where $\omega > 1$, $\tau > 1$, and $0 < \mu < \min\{\alpha, \lambda_c\}$, $\lambda_c + \mu < \frac{c}{2d}$.

Now we prove that for $i = 1, 2, \dots, m$, the left-hand side of (3.13) holds. Indeed, if $z > z_3$, we have $\underline{u_i}(z) = 0$. Since $\Gamma_i(z) \geq 0$ implies that $\Pi_i(z) \geq 0$. Therefore, we have $\underline{u_i}(z) \leq \Pi_i(z)$.

For $z \le z_3$, from (3.10) in Lemma 3.5, we have

$$\begin{aligned}
-d_i\underline{u_i}'' + c\underline{u_i}' + \xi_i\underline{u_i} &\le \xi_i\underline{u_i} + f_i(\underline{U}) - \delta\underline{u_i}^2 \\
&= \xi_i\underline{u_i} + \big(\sum_{j=1}^m a_{ij}\underline{u_j}\Big)\Big(b_{i0} + \sum_{k=m+1}^n b_{ik}\underline{u_k}\Big) - q_i\underline{u_i} - \delta\underline{u_i}^2 \\
&= [(\xi_i - q_i)\underline{u_i} - \delta\underline{u_i}^2] + \Big(\sum_{j=1}^m a_{ij}\underline{u_j}\Big)\Big(b_{i0} + \sum_{k=m+1}^n b_{ik}\underline{u_k}\Big).
\end{aligned}$$

Since $h(x) = (\xi_i - q_i)x - \delta x^2$ is increasing on $(0, \frac{\xi_i - q_i}{2\delta})$ and for $\delta > 0$ small enough, for $z \le z_3$,

$$\overline{u_i}(z) \le v_i e^{\lambda_c z} < \frac{\xi_i - q_i}{2\delta}.$$

Thus,

$$(\xi_i - q_i)\underline{u_i} - \delta\underline{u_i}^2 \le (\xi_i - q_i)u_i - \delta u_i^2.$$

Hence,

$$\begin{aligned}
-d_i\underline{u_i}'' + c\underline{u_i}' + \xi_i\underline{u_i} &\le [(\xi_i - q_i)u_i - \delta u_i^2] + \Big(\sum_{j=1}^m a_{ij}u_j\Big)\Big(b_{i0} + \sum_{k=m+1}^n b_{ik}u_k\Big) \\
&= \Gamma_i(U)(z).
\end{aligned}$$

It follows that

$$\begin{aligned}
\Pi_i(U)(z) &= \frac{1}{d_i x_i}\Big[\int_{-\infty}^z e^{x_{i1}(z-s)}\Gamma_i(U)(s)ds + \int_z^\infty e^{x_{i2}(z-s)}\Gamma_i(U)(s)ds\Big] \\
&\ge \frac{1}{d_i x_i}\Big[\int_{-\infty}^z e^{x_{i1}(z-s)} + \int_z^\infty e^{x_{i2}(z-s)}\Big][-d_i\underline{u_i}''(s) + c\underline{u_i}'(s) + \xi_i\underline{u_i}(s)]ds \\
&= \frac{1}{d_i x_i}\int_{-\infty}^z e^{x_{i1}(z-s)}[-d_i\underline{u_i}''(s) + c\underline{u_i}'(s) + \xi_i\underline{u_i}(s)]ds \\
&\quad + \int_z^{z_3} e^{x_{i2}(z-s)}[-d_i\underline{u_i}''(s) + c\underline{u_i}'(s) + \xi_i\underline{u_i}(s)]ds \\
&\quad + \int_{z_3}^\infty e^{x_{i2}(z-s)}[-d_i\underline{u_i}''(s) + c\underline{u_i}'(s) + \xi_i\underline{u_i}(s)]ds \\
&= \underline{u_i}(z) + \frac{1}{x_i}e^{x_{i2}(z-z_3)}\big[\underline{u_i}'(z_3+0) - \underline{u_i}'(z_3-0)\big] \\
&\ge \underline{u_i}(z).
\end{aligned}$$

Thus, we have proved that, for $i = 1, 2, \dots, m$, the left-hand side of (3.13) holds. Next we prove that, for $i = 1, 2, \dots, m$, the right-hand side of (3.13) holds. From Lemma 3.3, we have

$$\begin{aligned}
-d_i\overline{u_i}'' + c\overline{u_i}' + \xi_i\overline{u_i} &\ge \xi_i\overline{u_i} + f_i(\overline{U}) - \delta\overline{u_i}^2 \\
&= \xi_i\overline{u_i} + \Big(\sum_{j=1}^m a_{ij}\overline{u_j}\Big)(b_{i0} + \sum_{k=m+1}^n b_{ik}\overline{u_k}) - q_i\overline{u_i} - \delta\overline{u_i}^2 \\
&= [(\xi_i - q_i)\overline{u_i} - \delta\overline{u_i}^2] + \Big(\sum_{j=1}^m a_{ij}\overline{u_j}\Big)\Big(b_{i0} + \sum_{k=m+1}^n b_{ik}\overline{u_k}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\geq [(\xi_i - q_i)u_i - \delta u_i^2] + \Big(\sum_{j=1}^m a_{ij}u_j\Big)\Big(b_{i0} + \sum_{k=m+1}^n b_{ik}u_k\Big)\\
&= \Gamma_i(U)(z).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\Pi_i(U)(z) &= \frac{1}{d_i x_i}\Big[\int_{-\infty}^z e^{x_{i1}(z-s)}\Gamma_i(U)(s)ds + \int_z^\infty e^{x_{i2}(z-s)}\Gamma_i(U)(s)ds\Big]\\
&\leq \frac{1}{d_i x_i}\Big[\int_{-\infty}^z e^{x_{i1}(z-s)} + \int_z^\infty e^{x_{i2}(z-s)}\Big][-d_i\overline{u_i}''(s) + c\overline{u_i}'(s) + \xi_i\overline{u_i}(s)]ds\\
&= \frac{1}{d_i x_i}\int_{-\infty}^z e^{x_{i1}(z-s)}[-d_i\overline{u_i}''(s) + c\overline{u_i}'(s) + \xi_i\overline{u_i}(s)]ds\\
&\quad + \int_z^\infty e^{x_{i2}(z-s)}[-d_i\overline{u_i}''(s) + c\overline{u_i}'(s) + \xi_i\overline{u_i}(s)]ds = \overline{u_i}(z).
\end{aligned}
$$

Similarly, using the definitions of $\underline{U_S}(z)$ and $\overline{U_S}(z)$ and Lemma 3.4, we can prove that for $i = m+1, m+2, \dots, n$, (3.13) holds. Therefore, $\Pi(\Sigma) \subset \Sigma$.

Next we prove that the operator Π is continuous with respect to the norm $|\cdot|_\sigma$ in $B_\sigma(\mathbb{R}, \mathbb{R}^n)$. To do this, first, we take $\sigma > 0$ such that $0 < \sigma < \min\{-x_{i1}, x_{i2}, i = 1, 2, \dots n\}$.

For $U = (u_1(z), u_2(z), \dots, u_n(z)) \in \Sigma$ and $W = (w_1(z), w_2(z), \dots, w_n(z)) \in \Sigma$ and $i = 1, 2, \dots, m$,

$$
\begin{aligned}
&|\Gamma_i(U)(z) - \Gamma_i(W)(z)|e^{-\sigma|z|}\\
&= |\xi_i u_i + f_i(U) - \delta u_i^2 - \xi_i w_i - f_i(W) + \delta w_i^2|e^{-\sigma|z|}\\
&\leq [\xi_i|u_i - w_i| + \delta|u_i - w_i||u_i + w_i| + |f_i(U) - f_i(W)|]e^{-\sigma|z|}\\
&\leq [\xi_i|u_i - w_i| + \delta|u_i - w_i||u_i + w_i| + q_i|u_i - w_i|\\
&\quad + \Big|\Big(\sum_{j=1}^m a_{ij}u_j\Big)\Big(b_{i0} + \sum_{k=m+1}^n b_{ik}u_k\Big) - \Big(\sum_{j=1}^m a_{ij}w_j\Big)\Big(b_{i0} + \sum_{k=m+1}^n b_{ik}w_k\Big)\Big|\Big]e^{-\sigma|z|}\\
&= [(\xi_i + q_i)|u_i - w_i| + \delta|u_i - w_i||u_i + w_i|\\
&\quad + \Big|\Big(\sum_{j=1}^m a_{ij}u_j\Big)\Big(b_{i0} + \sum_{k=m+1}^n b_{ik}u_k\Big) - \Big(\sum_{j=1}^m a_{ij}w_j\Big)\Big(b_{i0} + \sum_{k=m+1}^n b_{ik}u_k\Big)\\
&\quad + \Big(\sum_{j=1}^m a_{ij}w_j\Big)\Big(b_{i0} + \sum_{k=m+1}^n b_{ik}u_k\Big) - \Big(\sum_{j=1}^m a_{ij}w_j\Big)\Big(b_{i0} + \sum_{k=m+1}^n b_{ik}w_k\Big)\Big|\Big]e^{-\sigma|z|}\\
&\leq \Big[(\xi_i + q_i)|u_i - w_i| + \delta|u_i - w_i||u_i + w_i|\\
&\quad + \Big|\Big(\sum_{j=1}^m a_{ij}(u_j - w_j)\Big|\Big|\Big(b_{i0} + \sum_{k=m+1}^n b_{ik}u_k\Big)\Big)\Big|\\
&\quad + \Big|\sum_{j=1}^m a_{ij}w_j\Big|\Big|\sum_{k=m+1}^n b_{ik}(u_k - w_k)|\Big]e^{-\sigma|z|}\\
&\leq L_i|U(\cdot) - W(\cdot)|_\sigma,
\end{aligned}
$$

where L_i is a positive constant depending on Q_i, q_i, a_{ij}, b_{ik}, v_i, ξ_i, and the constants in $\overline{u_i}$ and $\underline{u_i}$, $(i,j=1,2,\dots,m,\ k=m+1,\dots,n)$. Then

$$\begin{aligned}
&|\Pi_i(U)(z)-\Pi_i(W)(z)|e^{-\sigma|z|}\\
&\le \frac{e^{-\sigma|z|}}{d_ix_i}\Big[\int_{-\infty}^{z} e^{x_{i1}(z-s)}+\int_z^\infty e^{x_{i2}(z-s)}\Big]|\Gamma_i(U)(z)-\Gamma_i(W)(z)|ds\\
&\le \frac{L_ie^{-\sigma|z|}}{d_ix_i}\Big[\int_{-\infty}^{z} e^{x_{i1}(z-s)+\sigma|s|}\,ds+\int_z^\infty e^{x_{i2}(z-s)+\sigma|s|}\,ds\Big]|U(\cdot)-W(\cdot)|_\sigma\\
&= \frac{L_i}{d_ix_i}\Big[\int_{-\infty}^{z} e^{x_{i1}(z-s)-\sigma|z|+\sigma|s|}\,ds+\int_z^\infty e^{x_{i2}(z-s)-\sigma|z|+\sigma|s|}\,ds\Big]|U(\cdot)-W(\cdot)|_\sigma\\
&\le \frac{L_i}{d_ix_i}\Big[\int_{-\infty}^{z} e^{x_{i1}(z-s)+\sigma|z-s|}\,ds+\int_z^\infty e^{x_{i2}(z-s)+\sigma|z-s|}\,ds\Big]|U(\cdot)-W(\cdot)|_\sigma\\
&\le G_i|U(\cdot)-W(\cdot)|_\sigma,
\end{aligned}$$

where

$$G_i=\frac{L_i(x_{i1}-x_{i2}+2\sigma)}{d_ix_i(x_{i1}+\sigma)(x_{i2}-\sigma)}>0.$$

This implies

$$|\Pi_i(U)(\cdot)-\Pi_i(W)(\cdot)|_\sigma\le G_i|U(\cdot)-W(\cdot)|_\sigma.$$

Hence, $\Pi_i:\Sigma\to C(\mathbb{R},\mathbb{R})$ is continuous with respect to the norm $|\cdot|_\sigma$ in $B_\sigma(\mathbb{R},\mathbb{R}^n)$. Similarly, we can prove that, for $i=m+1,m+2,\dots,n$, $\Pi_i:\Sigma\to C(\mathbb{R},\mathbb{R})$ is also continuous with respect to the norm $|\cdot|_\sigma$ in $B_\sigma(\mathbb{R},\mathbb{R}^n)$. Therefore, Π: $\Sigma\to\Sigma$ is continuous with respect to the norm $|\cdot|_\sigma$ in $B_\sigma(\mathbb{R},\mathbb{R}^n)$.

Finally, we show that the operator Π is compact with respect to the norm $|\cdot|_\sigma$ in $B_\sigma(\mathbb{R},\mathbb{R}^n)$. Observe that, for $U(z)=(u_1(z),u_2(z),\dots,u_n(z))\in\Sigma$, we have

$$0\le u_i(z)\le N_i,$$

where N_i depends on v_i, λ_c, ω, τ, η, α, μ, q_i, Q_i. It follows that there is an M_i depending on these parameters as well as ξ_i such that

$$0\le\Gamma_i(U)\le M_i.$$

Thus,

$$\begin{aligned}
\Big|\frac{d}{dz}\Pi_i(U)(z)\Big|&=\frac{1}{d_ix_i}\Big|\Big[x_{i1}\int_{-\infty}^{z} e^{x_{i1}(z-s)}ds+x_{i2}\int_z^\infty e^{x_{i2}(z-s)}ds\Big]\Gamma_i(U)(s)\Big|\\
&\le\frac{M_i}{d_ix_i}\Big[|x_{i1}|\int_{-\infty}^{z} e^{x_{i1}(z-s)}ds+x_{i2}\int_z^\infty e^{x_{i2}(z-s)}ds\Big]\\
&=\frac{2M_i}{d_ix_i},
\end{aligned}$$

which implies

$$\Big|\frac{d}{dz}\Pi_i(U)(z)\Big|_\sigma\le\frac{2M_i}{d_ix_i}.$$

That is, $|\frac{d}{dz}\Pi_i(U)(z)|_\sigma$ is bounded. This is true for all $i=1,2,\dots,n$. This means that $\Pi(\Sigma)$ is uniformly bounded and equicontinuous with respect to the norm $|\cdot|_\sigma$.

For fixed positive integer k, define an operator $\Pi^k = (\Pi_1^k, \Pi_2^k, \dots, \Pi_n^k)$ by

$$\Pi^k(U)(z) = \begin{cases} \Pi(U)(-k) & z \in (-\infty, -k], \\ \Pi(U)(z) & z \in [-k, k], \\ \Pi(U)(k) & z \in [k, \infty). \end{cases}$$

By Arzela-Ascoli theorem, $\Pi^k : \Sigma \to \Sigma$ is compact with respect to the norm $|\cdot|_\sigma$ in $B_\sigma(\mathbb{R}, \mathbb{R}^n)$.

For $i = 1, 2, \dots, n$, we have

$$\begin{aligned} |\Pi_i(U)(z)| &= \frac{1}{d_i x_i}\Big|\Big[\int_{-\infty}^{z} e^{x_{i1}(z-s)} ds + \int_{z}^{\infty} e^{x_{i2}(z-s)} ds\Big] \Gamma_i(U)(s)\Big| \\ &\le \frac{M_i}{d_i x_i}\Big[\int_{-\infty}^{z} e^{x_{i1}(z-s)} ds + \int_{z}^{\infty} e^{x_{i2}(z-s)} ds\Big] \\ &= \frac{M_i}{d_i |x_{i1} x_{i2}|}. \end{aligned}$$

Therefore,

$$\begin{aligned} |\Pi_i^k(U)(\cdot) - \Pi_i(U)(\cdot)\big|_\sigma &= \sup_{z \in \mathbb{R}} \big|\Pi_i^k(U)(z) - \Pi_i(U)(z)\big| \, e^{-\sigma|z|} \\ &= \sup_{|z| \ge k} |\Pi_i^k(U)(z) - \Pi_i(U)(z)| e^{-\sigma|z|} \\ &\le \frac{2M_i}{d_i |x_{i1} x_{i2}|} e^{-\sigma k} \to 0 \end{aligned}$$

as $k \to \infty$. This means that

$$|\Pi^k(U)(\cdot) - \Pi(U)(\cdot)|_\sigma \to 0$$

as $k \to \infty$. That is, Π^k converges to Π in Σ with respect to the norm $|\cdot|_\sigma$. Therefore, $\Pi : \Sigma \to \Sigma$ is compact with respect to the norm $|\cdot|_\sigma$ in $B_\sigma(\mathbb{R}, \mathbb{R}^n)$. $\square$

Now we are ready to prove the existence of semi-traveling wave solution for system (3.2)

Proposition 3.7. *If $R_0 > 1$, there exists $\tilde{c} > 0$ such that for any $c > \tilde{c}$, system* (3.2) *has a nonnegative bounded semi-traveling wave solution $U(z)$ satisfying*

$$\lim_{z \to -\infty} U(z) = E_0.$$

Proof. For $R_0 > 1$, from Lemma 3.2, there exists $\tilde{c} > 0$ such that for any $c > \tilde{c}$, there exists $\lambda_c \in (0, \frac{c}{2d})$ with $d = \max_{1 \le i \le m} d_i$ and $\mathbf{V}_c = (v_1, \dots, v_m)$ with $v_i > 0$ such that $\Phi_I = (v_1 e^{\lambda_c z}, \dots, v_m e^{\lambda_c z})$ satisfies (3.5). Then we can define

$$\begin{aligned} \overline{U}(z) &= (\overline{U_I}(z), \overline{U_S}(z)), \\ \underline{U}(z) &= (\underline{U_I}(z), \underline{U_S}(z)) \end{aligned}$$

so that they satisfy Lemmas 3.3-3.5. Then we define operator

$$\Pi(U)(z) = (\Pi_1(U)(z), \dots, \Pi_n(U)(z)) : \Sigma \to \Sigma$$

such that (3.12) holds. From Lemma 3.6, we know that $\Pi(\Sigma) \subset \Sigma$ and it is continuous and compact with respect to the norm $|\cdot|_\sigma$ in $B_\sigma(\mathbb{R}, \mathbb{R}^n)$. Therefore, from Shauder's fixed point theorem, Π has a fixed point $U(z) \in \Sigma$, which satisfies (3.3). It is easily seen that

$$\lim_{z\to-\infty} \overline{U}(z) = \lim_{z\to-\infty} \underline{U}(z) = E_0.$$

It follows that $\lim_{z\to-\infty} U(z) = E_0$. This completes the proof. $\square$

Next we prove the existence of semi-traveling wave solutions of (1.2).

Theorem 3.8. *If $R_0 > 1$, then there exists $\tilde{c} > 0$ such that for any $c > \tilde{c}$, system (1.2) has a nonnegative bounded semi-traveling solution $U(z) = (u_1(z), \dots, u_n(z))$ satisfying*

$$\lim_{z\to-\infty} U(z) = E_0.$$

If $b_{ii} > 0$ for $i = m+1, \dots, n$, and there is at least one $1 \le j \le m$ such that $a_{ij} > 0$, then

$$u_i(z) < u_i^0.$$

For $i = 1, \dots, m$, $u_i(z) > 0$ and

$$\lim_{z\to-\infty} u_i(z)e^{-\lambda_c z} = v_i, \quad \lim_{z\to-\infty} u_i'(z)e^{-\lambda_c z} = \lambda_c v_i. \tag{3.14}$$

Proof. For each positive integer k, in (3.3), set $\delta = \delta_k = \frac{1}{k}$, we have $\delta_k \downarrow 0$ as $k \to +\infty$. From Proposition 3.7, for each δ_k, (3.2) has a nonnegative bounded semi-traveling wave solution

$$U_k(z) = (u_{1k}(z), u_{2k}(z), \dots, u_{nk}(z)) \in \Sigma$$

satisfying $\lim_{z\to-\infty} U_k(z) = E_0$. From the proof of Lemma 3.6 and system (3.3), $\{U_k(z)\}_{k=1}^{\infty}$, $\{U_k'(z)\}_{k=1}^{\infty}$, and $\{U_k''(z)\}_{k=1}^{\infty}$ are equicontinuous and uniformly bounded in $\mathbb{R}$. By Arzela-Ascoli theorem, there exists a subsequence $\{\delta_{k_j}\}$ such that for some

$$U(z) = (U_1(z), U_2(z), \dots, U_n(z)) \in \Sigma,$$
$$U_{k_j}(z) \to U(z), \quad U_{k_j}'(z) \to U'(z), \quad U_{k_j}''(z) \to U''(z)$$

as $j \to \infty$. Since $U_{k_j}(z)$ is a solution of (3.3), by taking $k_j \to \infty$ so that $\delta_{k_j} \to 0$, we know that $U(z)$ satisfies the wave equations corresponding (1.2), that is,

$$\begin{aligned} cU_I' &= d_I U_I'' + f_I(U), \\ cU_S' &= d_S U_S'' + f_S(U). \end{aligned} \tag{3.15}$$

Thus, $U(z)$ is a nonnegative semi-traveling solution of (1.2) satisfying

$$\lim_{z\to-\infty} U(z) = E_0.$$

Next, we show that $u_i(z) > 0$ for $i = 1, 2, \dots m$. Since $\underline{u_i}(z) = \max\{v_i e^{\lambda_c z}(1 - \tau e^{\mu z}), 0\}$, for $z < z_3 = -\frac{\ln \tau}{\mu}$, $\underline{u_i}(z) > 0$, we have $u_i(z) \ge \underline{u_i}(z) > 0$. If there is a z_0 such that $u_i(z_0) = 0$, then there are a and b such that $a < z_3 < b$ and $z_0 \in (a,\ b)$. Thus, $u_i(z)$ attains its nonpositive minimum over $[a,\ b]$ at z_0. Notice that on $(a,\ b)$, $u_i(z)$ satisfies

$$-d_i u_i'' + c_i u_i' = f_i(U(z)),$$

or

$$-d_i u_i'' + c_i u_i' + q_i u_i = \Big(\sum_{j=1}^{m} a_{ij} u_i\Big)\Big(b_{i0} + \sum_{k=m+1}^{n} u_k\Big) \ge 0,$$

By the strong maximum principle for second order elliptic equations (see [7]), $u \equiv 0$ on $(a,\ b)$. This is a contradiction since $u_i(z) > 0$ for $z \in (a, z_3)$. Therefore, we must have $u_i(z) > 0$ for all $z \in \mathbb{R}$. For $i = m+1, m+2, \dots, n$, we already know that $0 \le u_i(z) \le u_i^0$. If, at some point z^*, $u_i(z^*) = u_i^0$, then $u_i(z)$ attains its positive maximum at z^*. Therefore, $u_i'(z^*) = 0$, $u_i''(z^*) \le 0$. But, from (3.15), we have

$$d_i u_i''(z^*) = \Big(\sum_{j=1}^m a_{ij} u_j(z^*)\Big)\Big(\sum_{k=m+1}^n b_{ik} u_k(z^*)\Big) \ge \Big(\sum_{j=1}^m a_{ij} u_j(z^*)\Big)(b_{ii} u_i(z^*)) > 0.$$

This is a contradiction. Therefore, we must have $u_i(z) < u_i^0$.

Finally, we prove (3.14). We know that the semi-traveling wave solution $U(z)$ satisfies

$$\underline{U}(z) \le U(z) \le \overline{U}(z).$$

For $i = 1, 2, \dots m$,

$$\overline{u_i}(z) = \begin{cases} v_i e^{\lambda_c z}, & z \le z_1 = \frac{\ln \omega}{\lambda_c}, \\ v_i \omega, & z > z_1, \end{cases}$$

and

$$\underline{u_i}(z) = \begin{cases} v_i e^{\lambda_c z}(1 - \tau e^{\mu z}), & z \le z_3 = -\frac{\ln \tau}{\mu}, \\ 0, & z > z_3, \end{cases}$$

where $\omega > 1$, $\tau > 1$, and $0 < \mu < \min\{\alpha, \lambda_c\}$, $\lambda_c + \mu < \frac{c}{2d}$. Therefore, for $z < 0$ with $|z|$ large enough, we have

$$v_i e^{\lambda_c z}(1 - \tau e^{\mu z}) \le u_i(z) \le v_i e^{\lambda_c z}.$$

That is,

$$v_i(1 - \tau e^{\mu z}) \le u_i(z) e^{-\lambda_c z} \le v_i.$$

Thus,

$$\lim_{z \to -\infty} u_i(z) e^{-\lambda_c z} = v_i.$$

To prove that

$$\lim_{z \to -\infty} u_i'(z) e^{-\lambda_c z} = \lambda_c v_i,$$

let us consider a nonnegative semi-traveling wave solution $U_k(z) = (U_{Ik}(z), U_{Sk}(z))$ of the auxiliary system (3.2) with $\delta = \delta_k = \frac{1}{k}$. Using that $U_k(z)$ is a fixed point of operator Π_k and the definition of Π, it is easy to prove that

$$\lim_{z \to -\infty} u_i'(z) = 0, \quad i = 1, 2, \dots, n.$$

For $i = 1, 2, \dots m$, integrating both sides of the i-th equation of (3.3) from $-\infty$ to z and using the facts that $u_i(-\infty) = u_i'(-\infty) = 0$ reveal that

$$d_i u_i'(z) = c u_i(z) - \int_{-\infty}^z f_i(U)(s) ds + \delta_k \int_{-\infty}^z u_i^2(s) ds.$$

Therefore,

$$\begin{aligned} &\lim_{z \to -\infty} u_i'(z) e^{-\lambda_c z} \\ &= \lim_{z \to -\infty} \frac{1}{d_i}\Big(c u_i(z) e^{-\lambda_c z} - e^{-\lambda_c z} \int_{-\infty}^z f_i(U)(s) ds + \delta_k e^{-\lambda_c z} \int_{-\infty}^z u_i^2(s) ds\Big) \\ &= \frac{1}{d_i}\Big(c v_i - \lim_{z \to -\infty} \frac{\int_{-\infty}^z f_i(U)(s) ds}{e^{\lambda_c z}} + \lim_{z \to -\infty} \frac{\delta_k \int_{-\infty}^z u_i^2(s) ds}{e^{\lambda_c z}}\Big) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{d_i}\Big(cv_i - \lim_{z\to-\infty}\frac{f_i(U)(z)}{\lambda_c e^{\lambda_c z}} + \lim_{z\to-\infty}\frac{\delta_k u_i^2(z)dz}{\lambda_c e^{\lambda_c z}}\Big)\\
&= \frac{1}{d_i}\Big(cv_i - \lim_{z\to-\infty}\frac{e^{-\lambda_c z} f_i(U)(z)}{\lambda_c} + \lim_{z\to-\infty}\frac{\delta_k e^{-\lambda_c z} u_i^2(z)dz}{\lambda_c}\Big)\\
&= \frac{1}{d_i}\Big(cv_i - \frac{(\sum_{j=1}^m a_{ij}v_j)(b_{i0} + \sum_{k=m+1}^n b_{ik}u_k^0) - q_i v_i}{\lambda_c}\Big)\\
&= \frac{\lambda_c d_i v_i}{d_i} = \lambda_c v_i
\end{aligned}
$$

In the last equality we used (3.6). This completes the proof. □

4. Nonexistence of traveling wave solutions

The existence of semi-traveling wave solutions starting from the disease-free equilibrium E_0 means that the disease will spread among the population. The sufficient conditions to have these semi-traveling wave solutions that we proved are $R_0 > 1$ and $c > \tilde{c}$ with $\tilde{c} > 0$ depending on parameters in the system, especially on the diffusion coefficients of those infected subclasses. This implies that the speed of population movement will affect the outbreak of the disease. The question is what condition will guarantee the nonexistence of semi- traveling wave solutions? For this we have the following result.

Theorem 4.1. *If $R_0 < 1$, then for any $c > 0$, model* (1.2) *has no nonnegative nontrivial bounded semi-traveling wave solution satisfying* (3.1).

Proof. If $R_0 < 1$, suppose that (1.2) has a nonnegative nontrivial bounded semi-traveling wave solution $U(z) = (u_1(z), u_2(z), \dots, u_n(z))$ satisfying (3.1). Without loss of generality, we can assume that $0 \le u_i(z) \le u_i^0$ for $i = m+1, \dots, n$. Otherwise, we can rescale the solution so that this is true. Since $R_0 = \rho(EQ^{-1}) = \rho(Q^{-1}E) < 1$, by Perron-Frobenius theorem, there exists a $\mathbf{W} = (w_1, w_2, \dots, w_m) \in \mathbb{R}^m$ with $w_i > 0$ $(i = 1, 2, \dots m)$ such that

$$(Q^{-1}E)\mathbf{W} = R_0\mathbf{W}. \tag{4.1}$$

Observe that, for $i = 1, 2, \dots m$, $u_i(z)$ satisfies

$$cu_i' = d_i u_i'' + f_i(U).$$

It can be written as

$$d_i u_i'' + cu_i' + q_i u_i = A_i(z)B_i(z) = \Big(\sum_{j=1}^m a_{ij}u_j\Big)\Big(b_{i0} + \sum_{k=m+1}^n b_{ik}u_k\Big). \tag{4.2}$$

Let $\lambda_{1,2}^i$ be the roots of $-d_i\lambda^2 + c\lambda + q_i = 0$, that is,

$$\lambda_{1,2}^i = \frac{c \pm \sqrt{c^2 + 4d_i q_i}}{2d_i},$$

then (4.2) can be converted to

$$u_i(z) = \frac{1}{\Lambda_i}\Big[\int_{-\infty}^z e^{\lambda_2^i(z-s)}H_i(s)\,ds + \int_z^\infty e^{\lambda_1^i(z-s)}H_i(s)\,ds\Big], \tag{4.3}$$

where $\Lambda_i = \lambda_1^i - \lambda_2^i > 0$ and $H_i(s) = A_i(s)B_i(s)$. From (4.3) and the assumption $0 \le u_i(z) \le u_i^0$, $i = m+1, m+2, \dots, n$, we obtain

$$\begin{aligned} 0 &\le u_i(z) \\ &= \frac{q_i}{\Lambda_i}\Big[\int_{-\infty}^{z} e^{\lambda_2^i(z-s)}\frac{1}{q_i}H_i(s)\,ds + \int_z^{\infty} e^{\lambda_1^i(z-s)}\frac{1}{q_i}H_i(s)\,ds\Big] \\ &\le (Q^{-1}E)_i\frac{q_i}{\Lambda_i}\Big[\int_{-\infty}^{z} e^{\lambda_2^i(z-s)}u_I(s)\,ds + \int_z^{\infty} e^{\lambda_1^i(z-s)}u_I(s)\,ds\Big], \end{aligned} \tag{4.4}$$

where $(Q^{-1}E)_i$ $(i = 1, 2, \dots, m)$ denotes the i-th row of the matrix $Q^{-1}E$ and $u_I(s) = (u_1(s), u_2(s), \dots, u_m(s))^T$.

Let $u_i^* = \sup_{z\in\mathbb{R}} u_i(z)$ for $i = 1, 2, \dots, m$. Then $u^* = (u_1^*, u_2^*, \dots, u_m^*)^T \ge 0$ and $u^* \ne 0$. By taking supremum on the both sides of (4.4), we obtain

$$u^* \le (Q^{-1}E)u^* \tag{4.5}$$

Since $\mathbf{W}$ is positive, there exists a positive constant $\nu > 0$ such that

$$u^* \le \nu\mathbf{W} \tag{4.6}$$

Using (4.5), (4.6), and (4.1) gives us

$$u^* \le (Q^{-1}E)\nu\mathbf{W} = \nu(Q^{-1}E)\mathbf{W} = \nu R_0\mathbf{W}. \tag{4.7}$$

Using (4.5), (4.7), and (4.1) leads to $u^* \le \nu R_0^2\mathbf{W}$. Continuing this process results in

$$0 \le u^* \le \nu R_0^m\mathbf{W}$$

for all positive integer m. Taking limit $m \to \infty$, since $R_0 < 1$, leads to $u^* = 0$. This is a contradiction. This completes the proof. $\square$

5. Discussion

To investigate the influence of the mobility of population on the spread of disease, we propose and study a very general epidemic model which inculdes a large class of epidemic models that possess a disease-free equilibrium. Inspired by some specific models, we introduced basic reproduction number R_0. It turns out that the basic reproduction number R_0 is a threshold in the sense that the disease vanishes if $R_0 < 1$ and spreads if $R_0 > 1$.

We introduced a critical wave speed $\tilde{c}$ and established the existence of semi-traveling wave solutions that start from the disease- free equilibrium for $c > \tilde{c} > 0$. In order to investigate the final state of the disease, we need to establish the existence of traveling wave solution. To do this, we need to impose more restrictions on the model to guarantee the existence of another nontrivial steady state equilibrium and the semi-traveling wave solutions whose existence we established also connect this steady state.

As we mentioned before, this model is neither cooperative nor competitive. For these models, as the one investigated in [19], the spreading speed may be greater than the minimal wave speed. In a future work, we will investigate the relationship among the critical wave speed, the minimal wave speed, and the spreading speed. This is to prove that the critical wave speed introduced here is the same as the minimal wave speed. Besides proving the existence of traveling wave solution for $c > \tilde{c}$, we also need to prove that, for $R_0 > 1$ and $c < \tilde{c}$, there is no traveling wave solution.

In our model, all the parameters are constants. If this is not the case, the behavior of the disease will be very different. This is another direction of a future work.

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