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CONVERGENCE THEOREMS OF IMPLICIT TYPE ITERATIONS IN GEODESIC SPACES WITH NEGATIVE CURVATURE

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Abstract. In this article, we prove convergence theorems of the implicit iterative methods in the sense of Browder type and Xu-Ori type with (-1)-convex combination in CAT(-1) spaces.

1. Introduction

In recent years, fixed point theory has been investigated by many mathematicians. In particular, approximating fixed points of a nonlinear mapping is one of the main topics in this theory. Researchers have investigated some types of approximating iteration to find a fixed point of a mapping in several spaces, such as Banach spaces and geodesic spaces.

This paper considers two types of iterative schemes: explicit type schemes and implicit type schemes. This research field utilizes explicit iteration types, particularly Halpern and Mann types. However, implicit type methods, like Browder [10] and Xu-Ori [11] types, also have their significance.

Explicit type schemes generate a sequence $\{x_n\}$ by explicitly expressing x_{n+1} in terms of x_n. Halpern and Mann types iteration are explicit type schemes to find a fixed point of a mapping $T\colon X \to X$. These define a sequence $\{x_n\}$ as follows:

- Halpern type: $x_{n+1} := \alpha_n u \oplus (1-\alpha_n)Tx_n$;
- Mann type: $x_{n+1} := \alpha_n x_n \oplus (1-\alpha_n)Tx_n$

for $n \in \mathbb{N}$. On the other hand, there are some implicit type schemes such as Browder type and Xu-Ori type. These generate a sequence $\{x_n\}$ by finding the unique element x_n satisfying the following equations:

- Browder type: $x_n = \alpha_n u \oplus (1-\alpha_n)Tx_n$;
- Xu-Ori type: $x_n = \alpha_n x_{n-1} \oplus (1-\alpha_n)Tx_n$

for $n \in \mathbb{N}$. In this article, we consider implicit type schemes in geodesic spaces, particularly complete CAT(-1) spaces.

Recently, Kimura [6] proved the following convergence theorem with multiple anchor points $\{u_k\}$ in a complete CAT(0) space (which is also known as a Hadamard space). It uses the Browder type iterative scheme for multiple anchor points.

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Theorem 1.1 (Kimura [6, Theorem 3.3]). *Let X be a Hadamard space and let $T\colon X \to X$ be a nonexpansive mapping such that $F(T) \neq \emptyset$, where $F(T)$ is a set of all fixed points of T. Suppose that $\{\alpha_n\} \subset]0,1[$ such that $\alpha_n \to 0$ as $n \to \infty$. For $k = 1, 2, \dots, r$, let $\{\beta_n^k\} \subset [0,1]$ such that $\sum_{k=1}^r \beta_n^k = 1$ for all $n \in \mathbb{N}$ and $\beta_n^k \to \beta^k \in [0,1]$ as $n \to \infty$. Let $u_1, u_2, \dots, u_r \in X$ and define $\{x_n\} \subset X$ by*

$$x_n = \operatorname{argmin}_{y \in X} \Big(\alpha_n \sum_{k=1}^{r} \beta_n^k d(y, u_k)^2 + (1 - \alpha_n) d(y, Tx_n)^2 \Big)$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges to the unique minimizer of a function $g\colon F(T) \to \mathbb{R}$ defined by

$$g(y) = \sum_{k=1}^{r} \beta^k d(y, u_k)^2$$

for $y \in F(T)$.

Furthermore, Kimura also proved the following Δ-convergence theorem with an implicit iterative scheme for a finite family of nonexpansive mappings by using the Xu-Ori type iterative scheme.

Theorem 1.2 (Kimura [7, Theorem 3.2]). *Let X be a Hadamard space. For $k = 1, 2, \dots, N$, let $T_k\colon X \to X$ be a nonexpansive mapping such that $\bigcap_{k=1}^N F(T_k) \neq \emptyset$. For $k = 0, 1, \dots, N$, suppose $\{\alpha_n^k\} \subset [a,b] \subset]0,1[$ such that $\sum_{k=0}^N \alpha_n^k = 1$. For given $x_1 \in X$, generate a sequence $\{x_n\} \subset X$ satisfying*

$$x_{n+1} = \operatorname{argmin}_{y \in X} \Big(\alpha_n^0 d(x_n, y)^2 + \sum_{k=1}^{N} \alpha_n^k d(T_k x_{n+1}, y)^2 \Big)$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ is well-defined and Δ-convergent to some $x_0 \in \bigcap_{k=1}^N F(T_k)$.

In this article, we prove convergence theorems for implicit iterative methods in the sense of Browder and Xu-Ori types with (-1)-convex combination in complete CAT(-1) spaces.

2. Preliminaries

Let (X, d) be a metric space. For $x, y \in X$ and $l \geq 0$, a mapping $c\colon [0, l] \to X$ is called a geodesic with endpoints $x, y \in X$ if it satisfies $c(0) = x$, $c(l) = y$, and $d(c(t), c(s)) = |t - s|$ for every $t, s \in [0, l]$. Then $l = d(c(0), c(l)) = d(x, y)$. We say X is a geodesic space if a geodesic with endpoints x and y exists for all $x, y \in X$. In this paper, we assume X has the unique geodesic for every $x, y \in X$. Then, we denote the image of the geodesic with endpoints $x, y \in X$ by $[x, y]$, which is well defined. We call $[x, y]$ a geodesic segment with endpoints x and y.

Let $\mathbb{E}^2$ be the 2-dimensional Euclidean space, and let $\mathbb{H}^2$ be the 2-dimensional hyperbolic space, which are both geodesic spaces. For $\kappa \leq 0$, let M_κ^2 be a 2-dimensional space with constant curvature κ defined by

$$M_\kappa^2 = \begin{cases} \mathbb{E}^2 & \text{if } \kappa = 0; \\ \frac{1}{\sqrt{-\kappa}} \mathbb{H}^2 & \text{if } \kappa < 0, \end{cases}$$

where $\frac{1}{\sqrt{-\kappa}} \mathbb{H}^2$ is a geodesic space defined from $\mathbb{H}^2$ by multiplying the metric on $\mathbb{H}^2$ by $1/\sqrt{-\kappa}$.

Let (X, d) be a geodesic space. For $x, y, z \in X$, a geodesic triangle $\triangle(x, y, z)$ is defined as the union of three segments $[x, y], [y, z]$, and $[z, x]$. Fix $\kappa \le 0$ and let M_κ^2 be a model space with a metric ρ_κ. For each geodesic triangle $\triangle(x, y, z)$ on X, its comparison triangle $\bar{\triangle}(\bar{x}, \bar{y}, \bar{z})$ is defined as the triangle in M_κ^2 whose length of each corresponding edge is identical with that of the original triangle:

$$d(x, y) = \rho_\kappa(\bar{x}, \bar{y}), \quad d(y, z) = \rho_\kappa(\bar{y}, \bar{z}), \quad d(z, x) = \rho_\kappa(\bar{z}, \bar{x}).$$

A point $\bar{p} \in \bar{\triangle}(\bar{x}, \bar{y}, \bar{z})$ is called a comparison point for $p \in \triangle(x, y, z)$ if $d(u, p) = \rho_\kappa(\bar{u}, \bar{p})$ and $d(v, p) = \rho_\kappa(\bar{v}, \bar{p})$, where u, v are adjacent endpoints of p. A geodesic space X is called a CAT(κ) space if for all triangles $\triangle(x, y, z)$, points $p, q \in \triangle(x, y, z)$, and their comparison points $\bar{p}, \bar{q} \in \bar{\triangle}(\bar{x}, \bar{y}, \bar{z})$, the inequality

$$d(p, q) \le \rho_\kappa(\bar{p}, \bar{q}) \tag{2.1}$$

holds. The inequality (2.1) is called the CAT(κ) inequality.

It is clear that the n-dimensional Euclidean space $(\mathbb{E}^n, d_E)$ is an example of the complete CAT(0) spaces, since it always satisfies $d_E(p, q) = \rho_0(\bar{p}, \bar{q})$ in (2.1). More generally, the class of complete CAT(0) spaces consists of the class of Hilbert spaces. A complete CAT(0) space is often called a Hadamard space. We know that a Banach space is not a CAT(0) space in general. Furthermore, the n-dimensional hyperbolic space $\mathbb{H}^n$ is a complete CAT(-1) space, but the n-dimensional Euclidean space $\mathbb{E}^n$ is not a CAT(-1) space.

Let (X, d) be a geodesic space. Then, for $x, y \in X$ and $t \in [0, 1]$, there exists the unique point $z \in [x, y]$ such that $d(x, z) = (1-t)d(x, y)$ and $d(z, y) = td(x, y)$. Such a point z is called a convex combination of x and y. We denote it by $tx \oplus (1 - t)y$.

Let (X, d) be a CAT(0) space and let $(\mathbb{E}^2, \rho)$ be the 2-dimensional Euclidean space. Let $\triangle(x, y, z)$ be a geodesic triangle on X and take its comparison triangle $\bar{\triangle}(\bar{x}, \bar{y}, \bar{z})$ on $\mathbb{E}^2$. Then we know that the following equation, known as Stewart's theorem, holds for all $t \in [0, 1]$:

$$\rho(\bar{z}, t\bar{x} \oplus (1 - t)\bar{y})^2 = t\rho(\bar{z}, \bar{x})^2 + (1 - t)\rho(\bar{z}, \bar{y})^2 - t(1 - t)\rho(\bar{x}, \bar{y})^2.$$

This can be obtained by the following calculation in $\mathbb{R}^2$:

$$\begin{aligned} \|\bar{z} - (t\bar{x} + (1 - t)\bar{y})\|^2 &= \langle \bar{z} - (t\bar{x} + (1 - t)\bar{y}), \bar{z} - (t\bar{x} + (1 - t)\bar{y})\rangle \\ &= t\|\bar{z} - \bar{x}\|^2 + (1 - t)\|\bar{z} - \bar{y}\|^2 - t(1 - t)\|\bar{x} - \bar{y}\|^2. \end{aligned}$$

Note that $\mathbb{R}^2$ is one of the models of 2-dimensional Euclidean space. Moreover, since X is a CAT(0) space, we have $d(z, tx \oplus (1 - t)y) \le \rho(\bar{z}, t\bar{x} \oplus (1 - t)\bar{y})$. Therefore, since $d(z, x) = \rho_\kappa(\bar{z}, \bar{x})$, $d(z, y) = \rho_\kappa(\bar{z}, \bar{y})$, and $d(x, y) = \rho_\kappa(\bar{x}, \bar{y})$, we obtain an inequality

$$d(z, tx \oplus (1 - t)y)^2 \le td(z, x)^2 + (1 - t)d(z, y)^2 - t(1 - t)d(x, y)^2 \tag{2.2}$$

for all $t \in [0, 1]$. We introduce the following characterization of CAT(0) spaces.

Theorem 2.1 ([1, Theorem 1.3.3])**.** *For a geodesic space (X, d), the following two conditions are equivalent:*

(a) (X, d) *is a* CAT(0) *space;*
(b) *the inequality* (2.2) *holds for all $x, y, z \in X$ and $t \in [0, 1]$.*

Similarly, the following inequality holds for every CAT(-1) space X:

$$\begin{aligned} &\cosh d(z, tx \oplus (1 - t)y) \sinh d(x, y) \\ &\le \cosh d(z, x) \sinh(td(x, y)) + \cosh d(z, y) \sinh((1 - t)d(x, y)) \end{aligned}$$

for every $x, y, z \in X$ and $t \in [0, 1]$. This is obtained by the following equation on the 2-dimensional hyperbolic space $(\mathbb{H}^2, \rho)$:

$$\begin{aligned}&\cosh \rho(\bar{z}, t\bar{x} \oplus (1-t)\bar{y}) \sinh \rho(\bar{x}, \bar{y})\\&= \cosh \rho(\bar{z}, \bar{x}) \sinh(t\rho(\bar{x}, \bar{y})) + \cosh \rho(\bar{z}, \bar{y}) \sinh((1-t)\rho(\bar{x}, \bar{y}))\end{aligned}$$

for every $\bar{x}, \bar{y}, \bar{z} \in \mathbb{H}^2$ and $t \in [0, 1]$.

We know that any CAT(κ) is a CAT(κ') for $\kappa < \kappa'$. Therefore, every results for CAT(0) spaces can apply to any CAT(κ) spaces with $\kappa \leq 0$. For more details, see [2].

Let X be a CAT(0) space. A subset C of X is said to be convex if $tx \oplus (1-t)y \in C$ for all $x, y \in C$ and $t \in]0, 1[$.

Let X be a Hadamard space, and let C be a nonempty closed convex subset of X. Then there exists the unique point $p_x \in C$ such that $d(x, p_x) = \inf_{y \in C} d(x, y)$ for each $x \in X$. We define the metric projection P_C from X onto C by $P_C x = p_x$ for all $x \in X$.

Let X be a CAT(0) space. For a bounded sequence $\{x_n\}$ in X, let $r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n)$ for $x \in X$, and define the asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ by

$$r(\{x_n\}) = \inf_{x \in X} r(x, \{x_n\}).$$

The asymptotic center $\mathrm{AC}(\{x_n\})$ of $\{x_n\}$ is a set of all points $p \in X$ such that

$$r(p, \{x_n\}) = r(\{x_n\}).$$

If a CAT(0) space X is complete, then an asymptotic center of a bounded sequence $\{x_n\}$ on X is unique, see [3, Proposition 7].

Let X be a CAT(0) space. We say a sequence $\{x_n\}$ on X is Δ-convergent to $x_0 \in X$ if x_0 is the unique element of the asymptotic center of any subsequence of $\{x_n\}$. Then x_0 is called a Δ-limit of $\{x_n\}$.

Theorem 2.2 (Kirk and Panyanak [8, Proposition 3.5])**.** *Let X be a Hadamard space and let $\{x_n\}$ be a bounded sequence on X. Then there exists a Δ-convergent subsequence of $\{x_n\}$.*

Let X be a CAT(0) space. A mapping $T\colon X \to X$ is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y)$$

for every $x, y \in X$. We know the set $F(T) = \{z \in X : z = Tz\}$ of all fixed points of a nonexpansive mapping T is closed and convex. A mapping $U\colon X \to X$ is called a contraction if there exists $\alpha \in [0, 1[$ such that for all $x, y \in X$,

$$d(Ux, Uy) \leq \alpha d(x, y).$$

If X is complete, then the Banach contraction principle guarantees the existence and uniqueness of a fixed point of U. Let f be a real function on X and let C be a nonempty subset of X. Then $\operatorname{argmin}_{x \in C} f(x)$ stands for the set of all minimizers of f on C. Furthermore, if $\operatorname{argmin}_{x \in C} f(x)$ consists of exactly one point, then $\operatorname{argmin}_{x \in C} f(x)$ directly denotes such a point.

In this article, we use the notion of (-1)-convex combination introduced by Kimura and Sasaki defined as follows:

Definition 2.3 (Kimura and Sasaki [9, Definition 3.6]). *Let X be a geodesic space. Then for all $u, v \in X$, and $\alpha \in [0,1]$, the set*

$$\operatorname{argmin}_{x\in X} (\alpha \cosh d(u,x) + (1-\alpha)\cosh d(v,x))$$

is a singleton. Thus define a (-1)-convex combination of u and v by

$$\alpha u \overset{-1}{\oplus} (1-\alpha)v := \operatorname{argmin}_{x\in X} (\alpha \cosh d(u,x) + (1-\alpha)\cosh d(v,x)).$$

We know that $\alpha u \overset{-1}{\oplus} (1-\alpha)v \in [u,v]$ for all $u, v \in X$ and $\alpha \in [0,1]$. Namely,

$$\alpha u \overset{-1}{\oplus} (1-\alpha)v = \operatorname{argmin}_{x\in [u,v]} (\alpha \cosh d(u,x) + (1-\alpha)\cosh d(v,x))$$

holds, see [9, Lemma 3.5].

Lemma 2.4 ([9]). *Let X be a geodesic space. For $x, y \in X$ with $x \neq y$ and $\alpha \in [0,1]$, an equation*

$$\alpha x \overset{-1}{\oplus} (1-\alpha)y = \sigma x \oplus (1-\sigma)y$$

holds, where

$$\sigma = \frac{1}{d(x,y)} \tanh^{-1} \frac{\alpha \sinh d(x,y)}{1-\alpha+\alpha\cosh d(x,y)}.$$

It is obvious that $\alpha x \overset{-1}{\oplus} (1-\alpha)y = \alpha x \oplus (1-\alpha)y$ if $x = y$.

Lemma 2.5 ([9, Corollary 3.9]). *Let X be a* $\mathrm{CAT}(-1)$ *space and $x, y, z \in X$. Then for all $\alpha \in [0,1]$,*

$$\cosh d(\alpha x \overset{-1}{\oplus} (1-\alpha)y, z) \le \alpha \cosh d(x,z) + (1-\alpha)\cosh d(y,z).$$

Lemma 2.6 ([9, Lemma 3.7]). *For any $d > 0$ and $\alpha \in [0,1]$,*

$$\frac{1}{d}\tanh^{-1} \frac{\alpha \sinh d}{1-\alpha+\alpha\cosh d} + \frac{1}{d}\tanh^{-1} \frac{(1-\alpha)\sinh d}{\alpha+(1-\alpha)\cosh d} = 1.$$

Lemma 2.7 ([9, Lemma 3.4]). *For fixed $d > 0$ and $\alpha \in [0,1]$, let*

$$\sigma = \frac{1}{d}\tanh^{-1} \frac{\alpha \sinh d}{1-\alpha+\alpha\cosh d}.$$

Define a function $g\colon [0,1] \to \mathbb{R}$ by

$$g(t) = \alpha \cosh((1-t)d) + (1-\alpha)\cosh td$$

for $t \in [0,1]$. Then g is strictly convex and infinitely differentiable. Moreover, $g'(\sigma) = 0$ holds and hence σ is the unique minimizer of g.

The following results play important roles in the main results.

Theorem 2.8 (He, Fang, Lopez and Li [4, Proposition 2.3]). *Let X be a Hadamard space and $\{x_n\}$ a bounded sequence on X such that $x_n \overset{\Delta}{\rightharpoonup} x \in X$. Then, for all $u \in X$, the following holds:*

$$d(u,x) \le \liminf_{n\to\infty} d(u,x_n).$$

Lemma 2.9 (Kimura [5, Lemma 3.1]). *Let $\{x_n\}$ be a Δ-convergent sequence in a Hadamard space X with its Δ-limit $x \in X$. If $\{d(x_n,u)\}$ converges for some $u \in X$, then $\{x_n\}$ converges to x.*

3. Main results

In this section, we prove a convergence theorem with Browder and Xu-Ori type iteration in complete CAT(-1) spaces, respectively. To prove our main result, we first show the following lemmas.

Lemma 3.1. *Let $\alpha \in [0,1[$ and define a function $f\colon [0,\infty[\to \mathbb{R}$ by*

$$f(x) = x \tanh^{-1} \frac{(1-\alpha)\sinh x}{\alpha + (1-\alpha)\cosh x}$$

for $x \in \mathbb{R}$. Then, f is strictly increasing.

Proof. Fix $\alpha \in [0,1[$ and define $f_1\colon \mathbb{R} \to]-1,1[$ by

$$f_1(x) = \frac{(1-\alpha)\sinh x}{\alpha + (1-\alpha)\cosh x}$$

for $x \in \mathbb{R}$. Then

$$f_1'(x) = \frac{(1-a)(1+a(\cosh x - 1))}{(1+(1-a)(\cosh x - 1))^2} > 0$$

for all $x \in \mathbb{R}$ and hence f_1 is strictly increasing. Thus a function $f_2\colon [0,\infty[\to [0,\infty[$ defined by $f_2(x) = \tanh^{-1}(f_1(x))$ for $x \in [0,\infty[$ is also strictly increasing. This follows the strict increasingness of f. □

Lemma 3.2. *Let $d > 0$. Define $f\colon]0,\infty[\to \mathbb{R}$, by*

$$f(t) = \frac{\sinh td}{t}$$

for $t \in]0,\infty[$. Then, f is strictly increasing.

Proof. We have

$$f'(t) = \frac{td\cosh td - \sinh td}{t^2} = \frac{1}{t^2}\int_0^{td} x\sinh x\,dx > 0.$$

Thus we obtain the desired result. □

Lemma 3.3. *For fixed $d > 0$ and $\alpha \in]0,1/2[$, let*

$$\sigma = \frac{1}{d}\tanh^{-1}\frac{\alpha\sinh d}{1-\alpha+\alpha\cosh d}.$$

Then $\alpha < \sigma < 1/2$.

Proof. Define $g\colon [0,1] \to \mathbb{R}$ by

$$g(t) = \alpha\cosh((1-t)d) + (1-\alpha)\cosh td$$

for $t \in [0,1]$. Then σ is the unique minimizer of g from Lemma 2.7. Moreover, from the strict convexity of g, we have $g'(x) < 0$ for all $x \in]0,\sigma[$, and $g'(x) > 0$ for all $x \in]\sigma,1[$. Since $g'(1/2) = d(1-2\alpha)\sinh(d/2) > 0$, we have $\sigma < 1/2$. By $\alpha < 1/2 < 1-\alpha$ and Lemma 3.2, we have

$$\begin{aligned} g'(\alpha) &= -\alpha d\sinh((1-\alpha)d) + (1-\alpha)d\sinh\alpha d \\ &= d\alpha(1-\alpha)\Big(-\frac{\sinh((1-\alpha)d)}{1-\alpha} + \frac{\sinh\alpha d}{\alpha}\Big) < 0. \end{aligned}$$

Therefore, $\alpha < \sigma$. This is the desired result. □

Lemma 3.4. *For fixed $d > 0$, assume that $\alpha, \sigma \in [0, 1]$ satisfy the equation*
$$\sigma = \frac{1}{d} \tanh^{-1} \frac{\alpha \sinh d}{1 - \alpha + \alpha \cosh d}.$$
Then, $\alpha = 1/2$ if and only if $\sigma = 1/2$.

Proof. The given equation is equivalent to
$$\alpha = \frac{\sinh \sigma d}{\sinh \sigma d + \sinh((1 - \sigma)d)}.$$
From this we derives the conclusion using basic calculations. □

Lemma 3.5. *For fixed $d_1, d_2 \geq 0$ and $\alpha \in]0, 1/2[$, let*
$$\sigma_1 = \begin{cases} \frac{1}{d_1} \tanh^{-1} \frac{\alpha \sinh d_1}{1 - \alpha + \alpha \cosh d_1} & \text{if } d_1 \neq 0; \\ \alpha & \text{if } d_1 = 0 \end{cases}$$
and
$$\sigma_2 = \begin{cases} \frac{1}{d_2} \tanh^{-1} \frac{\alpha \sinh d_2}{1 - \alpha + \alpha \cosh d_2} & \text{if } d_2 \neq 0; \\ \alpha & \text{if } d_2 = 0. \end{cases}$$
Then, $\sigma_1 > \sigma_2$ if and only if $d_1 > d_2$. Moreover, $\sigma_1 = \sigma_2$ if and only if $d_1 = d_2$.

Proof. We consider the following cases:

- (i) $d_1 = 0$ or $d_2 = 0$:
  - (a) $d_1 = 0$ and $d_2 = 0$;
  - (b) $d_1 \neq 0$ and $d_2 = 0$;
  - (c) $d_1 = 0$ and $d_2 \neq 0$,
- (ii) $d_1 \neq 0$ and $d_2 \neq 0$:
  - (d) $d_1 = d_2$;
  - (e) $d_1 \neq d_2$.

First, we consider case (i).
(a) If $d_1 = d_2 = 0$, then it is obvious that $\sigma_1 = \alpha = \sigma_2$.
(b) Suppose that $d_1 \neq 0$ and $d_2 = 0$. Then $d_1 > d_2$. Furthermore, from Lemma 3.3, we have $\sigma_1 > \alpha = \sigma_2$.
(c) Similar to (b), if $d_1 = 0$ and $d_2 \neq 0$, then $d_1 < d_2$ and $\sigma_1 = \alpha < \sigma_2$ from Lemma 3.3.

Next, consider the case (ii). We hereinafter suppose that $d_1 \neq 0$ and $d_2 \neq 0$. Define a function $g\colon [0, 1] \to \mathbb{R}$ by
$$g(t) = \alpha \cosh((1 - t)d_1) + (1 - \alpha) \cosh td_1$$
for $t \in [0, 1]$. Then from Lemma 2.7, σ_1 is the unique minimizer of g. This follows that $g'(\sigma_2) > 0$ if and only if $\sigma_1 < \sigma_2$, and $g'(\sigma_2) < 0$ if and only if $\sigma_1 > \sigma_2$. We also get $\alpha < \sigma_1 < 1/2$ and $\alpha < \sigma_2 < 1/2$ by Lemma 3.3. By the definition of σ_2, we obtain
$$\alpha = \frac{\sinh \sigma_2 d_2}{\sinh \sigma_2 d_2 + \sinh((1 - \sigma_2)d_2)}.$$
Therefore,
$$g(t) = \frac{\sinh \sigma_2 d_2 \cosh((1 - t)d_1) + \sinh((1 - \sigma_2)d_2) \cosh td_1}{\sinh \sigma_2 d_2 + \sinh((1 - \sigma_2)d_2)}$$

for all $t \in [0,1]$. It follows that

$$g'(t) = \frac{d_1(-\sinh \sigma_2 d_2 \sinh((1-t)d_1) + \sinh((1-\sigma_2)d_2) \sinh t d_1)}{\sinh \sigma_2 d_2 + \sinh((1-\sigma_2)d_2)}$$

for all $t \in]0,1[$. Put $C = d_1/(\sinh \sigma_2 d_2 + \sinh((1-\sigma_2)d_2)) > 0$. Then

$$g'(\sigma_2) = C \cdot (-\sinh \sigma_2 d_2 \sinh((1-\sigma_2)d_1) + \sinh((1-\sigma_2)d_2) \sinh \sigma_2 d_1).$$

Put $p = (d_1 + d_2)/2$, $q = (d_2 - d_1)/2$, and $k = 1 - 2\sigma_2$. Then $p > 0$, $|q| < p$, $0 < k < 1$, and

$$\begin{aligned} g'(\sigma_2) &= C \cdot \Big(-\sinh\Big((p+q)\Big(\frac{1}{2} - \frac{1}{2}k\Big)\Big) \sinh\Big((p-q)\Big(\frac{1}{2} + \frac{1}{2}k\Big)\Big) \\ &\quad + \sinh\Big((p+q)\Big(\frac{1}{2} + \frac{1}{2}k\Big)\Big) \sinh\Big((p-q)\Big(\frac{1}{2} - \frac{1}{2}k\Big)\Big)\Big) \\ &= \frac{1}{2}C \cdot (-\cosh(p-kq) + \cosh(-kp+q) + \cosh(p+kq) - \cosh(kp+q)) \\ &= C \cdot (-\sinh kp \sinh q + \sinh kq \sinh p) \\ &= C \sinh p \sinh q(-f(p) + f(q)), \end{aligned}$$

where we define $f\colon \mathbb{R} \to]0,k]$ by

$$f(x) = \begin{cases} \dfrac{\sinh kx}{\sinh x} & \text{if } x \neq 0; \\ k & \text{if } x = 0 \end{cases}$$

for $x \in \mathbb{R}$. Then f is a differentiable even function and it satisfies $f'(x) > 0$ for all $x < 0$, and $f'(x) < 0$ for all $x > 0$.

(d): Suppose that $d_1 = d_2$. Then we have $q = 0$ and hence $g'(\sigma_2) = 0$. It implies that $\sigma_1 = \sigma_2$.

(e): Suppose that $d_1 \neq d_2$. Then since $|q| < p$, we obtain $-f(p) + f(q) > 0$. Therefore, $g'(\sigma_2) > 0$ if and only if $q > 0$, that is, $d_2 - d_1 > 0$. In other words, if $d_1 < d_2$, then $\sigma_1 < \sigma_2$; if $d_1 > d_2$, then $\sigma_1 > \sigma_2$.

From (i) and (ii), conditions $\sigma_1 > \sigma_2$ and $d_1 > d_2$ are equivalent, and so are conditions $\sigma_1 = \sigma_2$ and $d_1 = d_2$. ☐

Let X be a CAT(0) space. Then as noted in the preliminaries, the following inequality holds for every $x, y, z \in X$ and $t \in]0,1[$:

$$d(tx \oplus (1-t)y, z)^2 \leq td(x,z)^2 + (1-t)d(y,z)^2 - t(1-t)d(x,y)^2.$$

Since every CAT(-1) space is a CAT(0) space, the above inequality also holds in CAT(-1) spaces.

Theorem 3.6. *Let X be a* CAT(-1) *space and let $T\colon X \to X$ be a nonexpansive mapping. Let $u \in X$ and $\alpha \in]0, \frac{1}{2}]$. Define $U\colon X \to X$ by*

$$Ux = \alpha u \overset{-1}{\oplus} (1-\alpha)Tx$$

for $x \in X$. Then, U is a contraction.

Proof. Let $x, y \in X$. If $d(Ux, Uy) = 0$, then obviously there exists $\beta \in [0,1[$ such that $d(Ux, Uy) \leq \beta d(x,y)$. Thus, we consider the case where $d(Ux, Uy) \neq 0$. Then from Lemma 2.4, we have

$$d(Ux, Uy)^2$$

$$
\begin{aligned}
&= d(\alpha u \overset{-1}{\oplus} (1-\alpha)Tx, \alpha u \overset{-1}{\oplus} (1-\alpha)Ty)^2 \\
&= d(\sigma_1 u \oplus (1-\sigma_1)Tx, \sigma_2 u \oplus (1-\sigma_2)Ty)^2 \\
&\le \sigma_1 d(u, \sigma_2 u \oplus (1-\sigma_2)Ty)^2 + (1-\sigma_1) d(Tx, \sigma_2 u \oplus (1-\sigma_2)Ty)^2 \\
&\quad - \sigma_1(1-\sigma_1) d(u, Tx)^2 \\
&\le \sigma_1 (1-\sigma_2)^2 d(u, Ty)^2 + (1-\sigma_1)(\sigma_2 d(u, Tx)^2 + (1-\sigma_2) d(Tx, Ty)^2 \\
&\quad - \sigma_2(1-\sigma_2) d(u, Ty)^2) - \sigma_1(1-\sigma_1) d(u, Tx)^2 \\
&= (\sigma_1 - \sigma_2)((1-\sigma_2) d(u, Ty)^2 - (1-\sigma_1) d(u, Tx)^2) + (1-\sigma_1)(1-\sigma_2) d(Tx, Ty)^2,
\end{aligned}
$$

where

$$
\sigma_1 = \begin{cases} \dfrac{1}{d(u,Tx)} \tanh^{-1} \dfrac{\alpha \sinh d(u,Tx)}{1-\alpha+\alpha\cosh d(u,Tx)} & \text{if } u \neq Tx; \\ \alpha & \text{if } u = Tx; \end{cases}
$$

$$
\sigma_2 = \begin{cases} \dfrac{1}{d(u,Ty)} \tanh^{-1} \dfrac{\alpha \sinh d(u,Ty)}{1-\alpha+\alpha\cosh d(u,Ty)} & \text{if } u \neq Ty; \\ \alpha & \text{if } u = Ty. \end{cases}
$$

We consider the following two cases: (i) $\sigma_1 \ge \sigma_2$, and (ii) $\sigma_2 \ge \sigma_1$.

First, we consider the case (i). From Lemma 2.6, we have

$$
1-\sigma_1 = \begin{cases} \dfrac{1}{d(u,Tx)} \tanh^{-1} \dfrac{(1-\alpha) \sinh d(u,Tx)}{\alpha+(1-\alpha)\cosh d(u,Tx)} & \text{if } u \neq Tx; \\ 1-\alpha & \text{if } u = Tx; \end{cases}
$$

$$
1-\sigma_2 = \begin{cases} \dfrac{1}{d(u,Ty)} \tanh^{-1} \dfrac{(1-\alpha) \sinh d(u,Ty)}{\alpha+(1-\alpha)\cosh d(u,Ty)} & \text{if } u \neq Ty; \\ 1-\alpha & \text{if } u = Ty. \end{cases}
$$

Therefore,

$$
(1-\sigma_1) d(u,Tx)^2 = d(u,Tx) \tanh^{-1} \frac{(1-\alpha)\sinh d(u,Tx)}{\alpha+(1-\alpha)\cosh d(u,Tx)}
$$

and

$$
(1-\sigma_2) d(u,Ty)^2 = d(u,Ty) \tanh^{-1} \frac{(1-\alpha)\sinh d(u,Ty)}{\alpha+(1-\alpha)\cosh d(u,Ty)}.
$$

Using Lemmas 3.5 and 3.1, we obtain

$$
(1-\sigma_2) d(u,Ty)^2 \le (1-\sigma_1) d(u,Tx)^2.
$$

Similarly, we consider the case (ii) and then we obtain

$$
(1-\sigma_1) d(u,Tx)^2 \le (1-\sigma_2) d(u,Ty)^2.
$$

Therefore, in both cases (i) and (ii), we have

$$
d(Ux,Uy)^2 \le (1-\sigma_1)(1-\sigma_2) d(Tx,Ty)^2.
$$

By Lemmas 3.3 and 3.4, we have $\sigma_1 \ge \alpha$ and $\sigma_2 \ge \alpha$. Thus

$$
(1-\sigma_1)(1-\sigma_2) \le (1-\alpha)^2,
$$

and it follows that

$$
d(Ux,Uy)^2 \le (1-\alpha)^2 d(Tx,Ty)^2 \le (1-\alpha)^2 d(x,y)^2.
$$

Therefore,

$$d(Ux, Uy) \leq (1-\alpha)d(x, y),$$

and hence U is a contraction. □

Henceforth, we consider implicit-type iterative schemes. Now we prove a convergence theorem using Browder type iteration in complete CAT(-1) spaces.

Theorem 3.7. *Let X be a complete* CAT(-1) *space, and let $T\colon X \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $u \in X$ and $\{\alpha_n\} \subset]0, \frac{1}{2}]$ such that $\alpha_n \to 0$ as $n \to \infty$. Define $\{x_n\} \subset X$ by*

$$x_n = \alpha_n u \overset{-1}{\oplus} (1-\alpha_n)Tx_n.$$

Then, $\{x_n\}$ is well-defined and convergent to $P_{F(T)}u$.

Proof. We know that Theorem 3.6 implies the well-definedness of x_n for every $n \in \mathbb{N}$. Let $p = P_{F(T)}u$. Then

$$d(p, u) = \inf_{y \in F(T)} d(y, u).$$

By Lemma 2.5, we have

$$\begin{aligned}\cosh d(x_n, p) &= \cosh d(\alpha_n u \overset{-1}{\oplus} (1-\alpha_n)Tx_n, p)\\ &\leq \alpha_n \cosh d(u, p) + (1-\alpha_n)\cosh d(Tx_n, p)\\ &\leq \alpha_n \cosh d(u, p) + (1-\alpha_n)\cosh d(x_n, p).\end{aligned}$$

for all $n \in \mathbb{N}$. Thus,

$$\cosh d(x_n, p) \leq \cosh d(u, p)$$

for all $n \in \mathbb{N}$ and hence we obtain

$$d(Tx_n, p) \leq d(x_n, p) \leq d(u, p)$$

for all $n \in \mathbb{N}$. It implies that $\{x_n\}$ and $\{Tx_n\}$ are bounded. Since

$$d(x_n, Tx_n) \leq d(x_n, p) + d(p, Tx_n),$$

we have $\{d(x_n, Tx_n)\}$ is also bounded.

Fix $n \in \mathbb{N}$ and put $D = d(x_n, p)$. From the definition of x_n, we have

$$\begin{aligned}&(\alpha_n \cosh d(x_n, u) + (1-\alpha_n)\cosh d(x_n, Tx_n))\sinh D\\ &\leq (\alpha_n \cosh d(tx_n \oplus (1-t)p, u) + (1-\alpha_n)\cosh d(tx_n \oplus (1-t)p, Tx_n))\sinh D\\ &\leq \alpha_n(\cosh d(x_n, u)\sinh tD + \cosh d(p, u)\sinh(1-t)D)\\ &\quad + (1-\alpha_n)(\cosh d(x_n, Tx_n)\sinh tD + \cosh d(p, Tx_n)\sinh(1-t)D)\\ &= (\alpha_n \cosh d(x_n, u) + (1-\alpha_n)\cosh d(x_n, Tx_n))\sinh tD\\ &\quad + (\alpha_n \cosh d(p, u) + (1-\alpha_n)\cosh d(p, Tx_n))\sinh(1-t)D\end{aligned}$$

for all $t \in]0, 1[$. Thus

$$\begin{aligned}&(\alpha_n \cosh d(x_n, u) + (1-\alpha_n)\cosh d(x_n, Tx_n))\frac{\sinh D - \sinh tD}{\sinh(1-t)D}\\ &\leq \alpha_n \cosh d(p, u) + (1-\alpha_n)\cosh d(p, Tx_n).\end{aligned}$$

Letting $t \to 1$, we obtain

$$(\alpha_n \cosh d(x_n, u) + (1-\alpha_n)\cosh d(x_n, Tx_n))\cosh d(x_n, p)$$

$$\begin{aligned}&\leq \alpha_n \cosh d(p,u) + (1-\alpha_n)\cosh d(Tx_n,p)\\&\leq \alpha_n \cosh d(p,u) + (1-\alpha_n)\cosh d(x_n,p).\end{aligned}$$

Therefore,

$$\alpha_n \cosh d(x_n,u) + (1-\alpha_n)\cosh d(x_n,Tx_n) \leq \alpha_n \frac{\cosh d(p,u)}{\cosh d(x_n,p)} + (1-\alpha_n). \quad (3.1)$$

Thus, since $\alpha_n \to 0$ and $\{x_n\}$ is bounded, we obtain $\limsup_{n\to\infty} \cosh d(x_n,Tx_n) \leq 1$ from (3.1), and hence we have

$$\lim_{n\to\infty} d(x_n,Tx_n) = 0.$$

From (3.1), we obtain

$$\begin{aligned}\alpha_n \cosh d(x_n,u) &\leq \alpha_n \cosh d(x_n,u) + (1-\alpha_n)(\cosh d(x_n,Tx_n) - 1)\\&\leq \alpha_n \frac{\cosh d(p,u)}{\cosh d(x_n,p)}\\&\leq \alpha_n \cosh d(p,u).\end{aligned}$$

Thus, we obtain $\cosh d(x_n,u) \leq \cosh d(p,u)$ and it follows that

$$d(x_n,u) \leq d(p,u) \quad (3.2)$$

for all $n \in \mathbb{N}$.

To show that $\{x_n\}$ converges to p, we prove that $\{x_n\}$ is Δ-convergent to p. Thus, we take a subsequence $\{x_{n_i}\} \subset \{x_n\}$ arbitrarily, and let v be an element of the asymptotic center of $\{x_{n_i}\}$. Then, taking subsequence repeatedly, we can find $\{x'_j\} \subset \{x_{n_i}\}$ such that

$$\lim_{j\to\infty} d(x'_j,p) = \limsup_{i\to\infty} d(x_{n_i},p) \quad (3.3)$$

and there exists $q \in X$ such that $x'_j \xrightarrow{\Delta} q$ from Theorem 2.2. Then $q \in \mathrm{AC}(\{x'_j\})$. We show $q = p$. Since T is nonexpansive, we have

$$\begin{aligned}\limsup_{j\to\infty} d(x'_j,Tq) &\leq \limsup_{j\to\infty}(d(x'_j,Tx'_j) + d(Tx'_j,Tq))\\&\leq \limsup_{j\to\infty} d(x'_j,Tx'_j) + \limsup_{j\to\infty} d(Tx'_j,Tq)\\&\leq \limsup_{j\to\infty} d(x'_j,q).\end{aligned}$$

From the uniqueness of the element of $\mathrm{AC}(\{x'_j\})$, we obtain $q \in F(T)$. By Theorem 2.8 and (3.2), we have

$$d(q,u) \leq \liminf_{j\to\infty} d(x'_j,u) \leq d(p,u).$$

Since p is the unique nearest point of u on $F(T)$, the above inequality implies that $q = p$ and $p \in \mathrm{AC}(\{x'_j\})$. From (3.3), we have

$$\limsup_{i\to\infty} d(x_{n_i},p) = \lim_{j\to\infty} d(x'_j,p) \leq \limsup_{j\to\infty} d(x'_j,v) \leq \limsup_{i\to\infty} d(x_{n_i},v).$$

Hence $p \in \mathrm{AC}(\{x_{n_i}\})$ and it implies that $v = p$. Since v is an asymptotic center of $\{x_{n_i}\} \subset \{x_n\}$, which is arbitrarily chosen, and it coincides with p, $\{x_n\}$ is Δ-convergent to p.

We finally show the convergence of $\{x_n\}$ to p. Since $\{x_n\}$ is Δ-convergent to p and from (3.2), we have

$$d(p,u) \le \liminf_{n\to\infty} d(x_n,u) \le \limsup_{n\to\infty} d(x_n,u) \le d(p,u),$$

and hence we obtain

$$\lim_{n\to\infty} d(x_n,u) = d(p,u).$$

Therefore, $x_n \to p$ by Lemma 2.9, which is the desired result. □

We obtain the convergence theorem in the sense of Browder type with (-1)-convex combination in a complete CAT(-1) space. Next, we consider the convergence theorem in the sense of Xu-Ori type iteration in the same space.

Theorem 3.8. *Let X be a complete* CAT(-1) *space and let $T\colon X \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{\alpha_n\} \subset \mathbb{R}$ and $a \in \mathbb{R}$ satisfies $0 < a \le \alpha_n \le \frac{1}{2}$ for all $n \in \mathbb{N}$. Let $x_1 \in X$ and generate $\{x_n\}$ as follows: For $n \in \mathbb{N}$ and given $x_n \in X$, let x_{n+1} be the unique point in X satisfying that*

$$x_{n+1} = \alpha_n x_n \overset{-1}{\oplus} (1-\alpha_n) T x_{n+1}.$$

Then, $\{x_n\}$ is well-defined and Δ-convergent to some $x_0 \in F(T)$.

Proof. Fix $n \in \mathbb{N}$ and define a mapping $V_n\colon X \to X$ by

$$V_n x = \operatorname{argmin}_{y\in X} \left(\alpha_n \cosh d(y,x_n) + (1-\alpha_n)\cosh d(y,Tx)\right)$$

for $x \in X$. In the same way as Theorem 3.6, we obtain V_n is a contraction and thus it has the unique fixed point $x_{n+1} \in X$. That is, it satisfies that

$$x_{n+1} = V_n x_{n+1} = \operatorname{argmin}_{y\in X} \left(\alpha_n \cosh d(y,x_n) + (1-\alpha_n)\cosh d(y,Tx_{n+1})\right),$$

and hence $\{x_n\}$ is well-defined.

Next, we show $\{x_n\}$ is Δ-convergent to some element in $F(T)$. Let $p \in F(T)$ and $t \in\,]0,1[$. Fix $n \in \mathbb{N}$ and put $D = d(x_{n+1},p)$. Then,

$$\begin{aligned}
&(\alpha_n \cosh d(x_n,x_{n+1}) + (1-\alpha_n)\cosh d(Tx_{n+1},x_{n+1}))\sinh D \\
&= (\alpha_n \cosh d(x_n,V_n x_{n+1}) + (1-\alpha_n)\cosh d(Tx_{n+1},V_n x_{n+1}))\sinh D \\
&\le \alpha_n \cosh d(x_n, tx_{n+1} \oplus (1-t)p)\sinh D \\
&\quad + (1-\alpha_n)\cosh d(Tx_{n+1}, tx_{n+1}\oplus(1-t)p)\sinh D \\
&\le \alpha_n(\cosh d(x_n,x_{n+1})\sinh tD + \cosh d(x_n,p)\sinh(1-t)D) \\
&\quad + (1-\alpha_n)(\cosh d(Tx_{n+1},x_{n+1})\sinh tD \\
&\qquad + \cosh d(Tx_{n+1},p)\sinh(1-t)D) \\
&= (\alpha_n \cosh d(x_n,x_{n+1}) + (1-\alpha_n)\cosh d(Tx_{n+1},x_{n+1}))\sinh tD \\
&\quad + (\alpha_n\cosh d(x_n,p) + (1-\alpha_n)\cosh d(Tx_{n+1},p))\sinh(1-t)D.
\end{aligned}$$

Thus

$$\begin{aligned}
&(\alpha_n \cosh d(x_n,x_{n+1}) + (1-\alpha_n)\cosh d(Tx_{n+1},x_{n+1}))\frac{\sinh D - \sinh tD}{\sinh(1-t)D} \\
&\le \alpha_n \cosh d(x_n,p) + (1-\alpha_n)\cosh d(Tx_{n+1},p).
\end{aligned}$$

Letting $t \to 1$, we obtain

$$(\alpha_n \cosh d(x_n,x_{n+1}) + (1-\alpha_n)\cosh d(Tx_{n+1},x_{n+1}))\cosh d(x_{n+1},p)$$

$$\begin{aligned}&\le \alpha_n \cosh d(x_n, p) + (1-\alpha_n)\cosh d(Tx_{n+1}, p)\\&\le \alpha_n \cosh d(x_n, p) + (1-\alpha_n)\cosh d(x_{n+1}, p).\end{aligned}$$

Hence we have

$$\cosh d(x_{n+1}, p) \le \alpha_n \cosh d(x_n, p) + (1-\alpha_n)\cosh d(x_{n+1}, p).$$

Therefore, since $\{\alpha_n\} \subset]0, \frac{1}{2}]$, we obtain

$$\cosh d(x_{n+1}, p) \le \cosh d(x_n, p).$$

This implies that the real sequence $\{d(x_n, p)\}$ is nonincreasing and bounded below. Thus there exists a limit

$$\lim_{n\to\infty} d(x_n, p) = c_p \in \mathbb{R}$$

and hence

$$\begin{aligned}1 &\le \alpha_n \cosh d(x_n, x_{n+1}) + (1-\alpha_n)\cosh d(Tx_{n+1}, x_{n+1})\\&\le (\alpha_n \cosh d(x_n, p) + (1-\alpha_n)\cosh d(x_{n+1}, p))\frac{1}{\cosh d(x_{n+1}, p)}\\&\le \frac{\alpha_n(\cosh d(x_n, p) - \cosh d(x_{n+1}, p))}{\cosh d(x_{n+1}, p)} + 1\\&\to 1\end{aligned}$$

as $n \to \infty$. This implies

$$\lim_{n\to\infty}(\alpha_n \cosh d(x_n, x_{n+1}) + (1-\alpha_n)\cosh d(Tx_{n+1}, x_{n+1})) = 1.$$

Then

$$\lim_{n\to\infty} \cosh d(x_n, x_{n+1}) = \lim_{n\to\infty} \cosh d(Tx_{n+1}, x_{n+1}) = 1.$$

Indeed, we assume $\{\cosh d(x_n, x_{n+1})\}$ does not converge to 1. Then there exist $\varepsilon > 0$ and a subsequence $\{\cosh d(x_{n_i}, x_{n_i+1})\}$ of $\{\cosh d(x_n, x_{n+1})\}$ such that $\cosh d(x_{n_i}, x_{n_i+1}) \ge 1 + \varepsilon$ for $i \in \mathbb{N}$. Furthermore, since $\{\alpha_{n_i}\} \subset [a, \frac{1}{2}]$, we may assume that $\alpha_{n_i} \to \alpha_0 \in [a, \frac{1}{2}]$ without loss of generality. Then we have

$$\begin{aligned}1 &= \lim_{i\to\infty}(\alpha_{n_i} \cosh d(x_{n_i}, x_{n_i+1}) + (1-\alpha_{n_i})\cosh d(Tx_{n_i}, x_{n_i+1}))\\&\ge \alpha_0 \liminf_{i\to\infty} \cosh d(x_{n_i}, x_{n_i+1}) + (1-\alpha_0)\liminf_{i\to\infty} \cosh d(Tx_{n_i+1}, x_{n_i+1})\\&\ge \alpha_0(1+\varepsilon) + (1-\alpha_0) = 1 + \alpha_0\varepsilon > 1.\end{aligned}$$

This is a contradiction. Thus we have $\lim_{n\to\infty} \cosh d(x_n, x_{n+1}) = 1$, and similarly we obtain $\lim_{n\to\infty} \cosh d(Tx_{n+1}, x_{n+1}) = 1$. Hence we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(Tx_{n+1}, x_{n+1}) = 0.$$

Let $x_0 \in X$ be the unique asymptotic center of a sequence $\{x_n\}$ and let $u \in X$ be an asymptotic center of any subsequence $\{x_{n_i}\}$ of $\{x_n\}$. We will show that $u = x_0$. From the definition of the asymptotic center, we have

$$\begin{aligned}r(\{x_{n_i}\}) &= \limsup_{i\to\infty} d(x_{n_i}, u)\\&\le \limsup_{i\to\infty} d(x_{n_i}, Tu)\\&\le \limsup_{i\to\infty}(d(x_{n_i}, Tx_{n_i}) + d(Tx_{n_i}, Tu))\\&= \limsup_{i\to\infty} d(Tx_{n_i}, Tu)\end{aligned}$$

$$\le \limsup_{i\to\infty} d(x_{n_i}, u) = r(\{x_{n_i}\}).$$

This implies $Tu \in \mathrm{AC}(\{x_{n_i}\})$. From the uniqueness of an asymptotic center, we obtain $u \in F(T)$. It follows that $\{d(x_n, u)\}$ is convergent to $c_u \in \mathbb{R}$. Therefore,

$$\begin{aligned} r(\{x_n\}) &= \limsup_{n\to\infty} d(x_n, x_0) \\ &\le \limsup_{n\to\infty} d(x_n, u) = c_u = \lim_{i\to\infty} d(x_{n_i}, u) \\ &\le \limsup_{i\to\infty} d(x_{n_i}, x_0) \\ &\le \limsup_{n\to\infty} d(x_n, x_0) = r(\{x_n\}). \end{aligned}$$

Thus $u \in \mathrm{AC}(\{x_n\})$. From the uniqueness of an asymptotic center, we obtain $u = x_0$. Hence, $\{x_n\}$ is Δ-convergent to $x_0 \in F(T)$. This is the desired result. $\square$

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