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ASYMPTOTIC STABILITY FOR HILFER-LIKE NABLA NONLINEAR FRACTIONAL DIFFERENCE EQUATIONS

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Abstract. This article examines the asymptotic stability of nonlinear fractional difference equations with a Hilfer-like nabla operator. The results for a Hilfer-type nabla fractional difference that contains Riemann-Liouville and Caputo nabla difference as a particular case. We use Picard's iteration and a fixed point theorem to obtain results on existence and uniqueness. To obtain the main results, we use linear a scalar fractional difference equality, discrete comparison principle, and basics of difference equations. We also present a Lyapunov second direct method for nonlinear discrete fractional systems. We also discus stability results with some numerical examples.

1. Introduction

Fractional calculus is a field within mathematical analysis that explores the derivatives and integrals of non-integer orders and their applications in numerous scientific domains, including engineering and economics. Fractional Calculus (FC) emerged as a brilliant concept conceived by Gottfried Leibnitz towards the close of the seventeenth century and also has an essential role in science and engineering fields [7, 15, 17, 18, 19, 22, 25]. Discrete fractional calculus is a branch of mathematics that extends the concepts of traditional calculus to deal with discrete-time systems or signals with fractional orders. Unlike classical calculus, which deals with integer-order derivatives and integrals, discrete fractional calculus allows for the analysis of systems or signals with non-integer order dynamics, providing a more comprehensive understanding of complex phenomena encountered in various fields such as signal processing, control theory, and time series analysis. In the case of discrete fractional calculus, we cannot use the $\epsilon - \delta$ definition directly to prove the stability i.e. we can not provide the results or numerical methods for proving stability in the nonlinear fractional difference equations. In 1892, Lyapunov constructed the Lyapunov stability criteria. Lyapunov provides a method which is useful for checking the stability of nonlinear systems [16, 21, 26]. In [8] authors introduced a general method for finding quadratic Lyapunov functions in the stability analysis of many continuous fractional order systems. The basic idea of discrete stability of fractional systems is discussed in book [6]. In [23, 24] authors defined a diamond ϑ-derivative which is the linear combination of the standard ∇ and Δ derivative on

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the time scale. In [2] Anderson and Onitsuka investigated the Lyapunov stability analysis of the equilibrium solution in the discrete homogeneous linear first-order diamond-alpha derivative equation.

$$\diamond_\vartheta \eta(\omega) = \lambda \eta(\omega), \quad \diamond_\vartheta \eta(\omega) = \vartheta \Delta_h \eta(\omega) + (1-\vartheta)\nabla_h \eta(\omega), \quad \vartheta \in [0,1],$$

where $\lambda \in \mathbb{C}$ and $I = [\omega_0, \infty]_{h\mathbb{Z}}$ and h is step-size $h \in \mathbb{R}$, $h\mathbb{Z} = \{hn : n \in \mathbb{Z}\}$. Atici and Eloe [3, 4, 9, 12], Miller and Ross [20] and Anastassiou [1] introduced Liouville and Caputo nabla fractional difference and defined several properties. In [27] Wu, Baleanu and Luo found a alternative method to investigate the stability of the following nonlinear equations

$$\nabla_m^\nu \eta(\omega) = \xi(\omega, \eta(\omega)), \quad \eta(m+1) = C, \quad 0 < \nu < 1, \quad \omega \in \mathbb{N}_{m+2},$$

where $\nabla_m^\nu \eta(\omega)$ is the Riemann-Liouville difference of $\eta(\omega)$. In [5] Wu, Baleanu and Luo analyze the stability of the following Caputo-like discrete fractional systems

$${}_h^C\Delta_m^\nu \eta(\omega) = \xi(\omega + \nu h, \eta(\omega + \nu h)), \quad 0 < \nu \le 1,$$

where ${}_h^C\Delta_m^\nu \eta(\omega)$ denotes the Caputo-like delta fractional h-difference of $\eta(\omega)$ on the sets of discrete time $(h\mathbb{N})_{m+(1-\nu)h} = \{m + (1-\nu)h, m + (2-\nu)h, \cdots, \}$. In [14] the authors introduced the Linear Hilfer nabla fractional difference which includes Riemann-Liouville and Caputo nabla difference as a particular cases and defined the asymptotical behaviour of the solution. Discrete Hilfer fractional difference is a mathematical concept used in the analysis of discrete-time signals or systems. It is an extension of the traditional difference operator to non-integer orders, allowing for the characterization of signals with fractional dynamics. In [14] Mohan and Gopal derive the Volterra summation equation to solve an initial value problem for the class of nonlinear Hilfer Nabla difference equations. The authors demonstrated the existence of stable solutions for the nonlinear Cauchy problem in [11] by utilizing the compact method and fixed point technique. In [13] the authors presented two nonlinear nabla variable-order models and proved their asymptotic stability. After this in the last several years researchers gives a vigorous theory of fractional calculus of a real variables. In this article we investigate the uniqueness and existence of the Hilfer-like fractional nonlinear discrete dynamical system in two ways: first, by the Picard iteration method, and second, by fixed point theorem, which utilizes the compactness property. After this we continue the work done by the authors in [14, 27, 5, 28] and discuss the Lyapunov theory for the asymptotic stability of the Hilfer-type nabla fractional difference equation. We consider the following nonlinear equation

$$\begin{aligned} (\nabla_m^{\vartheta,\rho} \eta)(\omega) &= \xi(\omega, \eta(\omega)), \omega \in \mathbb{N}_{m+1} \\ [(\nabla_m^{-(1-\gamma)} \eta)(\omega)]_{\omega=m} &= \eta(m) = \eta_0 \end{aligned} \tag{1.1}$$

where $0 < \vartheta \le 1$, $0 \le \rho \le 1$, $\gamma = \vartheta + \rho - \vartheta\rho$, $\xi : \mathbb{N}_m \times \mathbb{R} \to \mathbb{R}$ is the nonlinearity.

Motivations and contributions.

- Existence and uniqueness using Picard's iteration method and fixed point theorem.
- Linear scalar fractional difference inequalities are employed.
- The Lyapunov second direct method is suggested for analyzing nonlinear discrete fractional systems.

- To investigate the asymptotic stability properties of nonlinear fractional difference equations, numerical examples with simulation results are discussed using the well known Lyapunov direct method and Newton iteration method.

The plan for this article is follows: In Section 2, contains preliminaries on discrete fractional calculus. In section 3, we discuss some basics notation of Hilfer nabla fractional difference and their properties. In section 4, we provide some inequalities, discrete comparison principle and method for Lyapunov stability of the nonlinear fractional difference equation. In section 5, we discuss asymptotic stability with some numerical examples using Lyapunov direct method and Newton's iteration method.

2. Preliminaries

Let $\mathbb{R}$ be the set of real numbers, and $\mathbb{N}_m$ as the set $\{m, m+1, m+2, \cdots\}$ for any $m \in \mathbb{R}$. It is assumed that empty sums equate to 0 and empty products equate to 1.

Definition 2.1 ([14]). Consider μ belonging to the set of real numbers, excluding values in the set $\{\cdots, -2, -1\}$. The μ^{th} order nabla fractional Taylor monomial is expressed as follows:

$$\mathcal{H}_\mu(\omega, m) = \frac{(\omega - m)^{\bar{\mu}}}{\Gamma(\mu+1)} = \frac{\Gamma(\omega - m + \mu)}{\Gamma(\omega - m)\Gamma(\mu+1)},$$

provided the right-hand side exists. $\Gamma(\cdot)$ is the Euler gamma function. Let $\eta : \mathbb{N}_m \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The first order backward (nabla) difference of η is defined by $(\nabla\eta)(\omega) = \eta(\omega) - \eta(\omega - 1)$ for $\omega \in \mathbb{N}_{m+1}$, and the $N^{th}-$order nabla difference of η defined recursively by $(\nabla^N\eta)(\omega) = (\nabla(\nabla^{N-1}\eta))(\omega)$ for $\omega \in \mathbb{N}_{m+N}$.

Definition 2.2 ([14]). Consider a function $\eta : \mathbb{N}_m \to \mathbb{R}$ and let $\nu > 0$. The $\nu^{th}-$order nabla sum of η is defined as follows

$$(\nabla_m^{-\nu}\eta)(\omega) = \sum_{s=m}^{\omega} \mathcal{H}_{\nu-1}(\omega, \zeta(s))\eta(s), \quad \omega \in \mathbb{N}_m,$$

where $\zeta(s) = s - 1$.

Definition 2.3 ([14]). Given a function $\eta : \mathbb{N}_m \to \mathbb{R}$, $\nu > 0$ and select $N \in \mathbb{N}_1$ such that $N - 1 < \nu \le N$. The $\nu^{th}-$order Riemann-Liouville nabla difference of η is defined by

$$(\nabla_m^{\nu}\eta)(\omega) = (\nabla^N(\nabla_m^{-(N-\nu)}\eta))(\omega), \quad \omega \in \mathbb{N}_{m+N}.$$

Definition 2.4 ([14]). Consider a function $\eta : \mathbb{N}_{m-N} \to \mathbb{R}$, $\nu > 0$ and select $N \in \mathbb{N}_1$ such that $N - 1 < \nu \le N$. The $\nu^{th}-$order Caputo nabla difference of η is defined as follows

$$(\nabla_{*m}^{\nu}\eta)(\omega) = (\nabla_m^{-(N-\nu)}(\nabla^N\eta)(\omega)), \quad \omega \in \mathbb{N}_m.$$

Lemma 2.5 ([10, 14]). *Let $\eta : \mathbb{N}_m \to \mathbb{R}$, ν and μ both are greater than 0. In this case,*

$$(\nabla_m^{-\nu}\nabla_m^{-\mu}\eta)(\omega) = (\nabla_m^{-\nu-\mu}\eta)(\omega), \quad \omega \in \mathbb{N}_m, \tag{2.1}$$

$$(\nabla_{m+1}^{-\nu}\nabla\eta)(\omega) = (\nabla\nabla_m^{-\nu}\eta)(\omega) - \mathcal{H}_{\nu-1}(\omega, \zeta(m))\eta(m), \quad \omega \in \mathbb{N}_{m+1}. \tag{2.2}$$

Lemma 2.6 ([14, 10])**.** *The well-definedness of the following fractional nabla Taylor monomials is established.*
1. Consider $\nu > 0$ and $\mu \in \mathbb{R}$. In this scenario,
$$\nabla_m^{-\nu}\mathcal{H}_\nu(\omega,\zeta(m)) = \mathcal{H}_{\nu+\mu}(\omega,\zeta(m)), \quad \omega \in \mathbb{N}_m.$$
2. Consider $\nu, \mu \in \mathbb{R}$ and $\omega \in \mathbb{N}_1$ such that $N-1 < \nu \leq N$. Then
$$\nabla_m^{\nu}\mathcal{H}_\mu(\omega,\zeta(m)) = \mathcal{H}_{\nu-\mu}(\omega,\zeta(m)), \quad \omega \in \mathbb{N}_{m+N}.$$

3. Hilfer nabla fractional difference

First, we provide the Hilfer fractional derivative's nabla equivalent.

Definition 3.1 ([14, 10])**.** Consider the function $\eta : \mathbb{N}_m \to \mathbb{R}$, $0 \leq \rho \leq 1$, and choose $N \in \mathbb{N}_1$ such that $N-1 < \vartheta \leq N$. The $\vartheta^{th}-$order and $\rho^{th}-$type Hilfer nabla difference of η is defined by
$$((\nabla_m^{\vartheta,\rho})\eta)(\omega) = (\nabla_{m+N}^{-\rho(N-\vartheta)}\nabla^N\nabla_m^{-(1-\rho)(N-\vartheta)}\eta)(\omega), \quad \omega \in \mathbb{N}_{m+N}.$$
The type ρ enables continuous interpolation between the Riemann-Liouville case and the Caputo case respectively such that $\nabla_m^{\vartheta,0} \equiv \nabla_m^{\vartheta}$ and $\nabla_m^{\vartheta,1} \equiv \nabla_{*m}^{\vartheta}$.

Remark 3.2 ([14, 10])**.** A function η defined on $\mathbb{N}_m$ with values in $\mathbb{R}$, then
$$(\nabla_m^{-(1-\rho)(N-\vartheta)}\eta) : \mathbb{N}_m \to \mathbb{R},$$
implying that
$$(\nabla^N\nabla_m^{-(1-\rho)(N-\vartheta)}\eta) : \mathbb{N}_{m+N} \to \mathbb{R},$$
$$(\nabla_{m+N}^{-\rho(N-\vartheta)}\nabla^N\nabla_m^{-(1-\rho)(N-\vartheta)}\eta) : \mathbb{N}_{m+N} \to \mathbb{R}.$$
So, if $\eta : \mathbb{N}_m \to \mathbb{R}$, then $(\nabla_m^{\vartheta,\rho}) : \mathbb{N}_{m+N} \to \mathbb{R}$.

Proposition 3.3 ([14] Power Rule)**.** *Let $\nu \in \mathbb{R}$, $0 < \vartheta \leq 1$, $0 \leq \rho \leq 1$, and assume that the following fractional nabla Taylor monomials are well defined. Then*
$$\nabla_m^{\vartheta,\rho}\mathcal{H}_\mu(\omega,\zeta(m)) = \mathcal{H}_{\mu-\vartheta}(\omega,\zeta(m)) - \mathcal{H}_{\rho(1-\vartheta)-1}(\omega,\zeta(m)), \quad \omega \in \mathbb{N}_{m+1}.$$

Proposition 3.4 ([14] Composition Rule)**.** *Suppose $\eta : \mathbb{N}_m \to \mathbb{R}$, where $0 < \vartheta \leq 1$, $\quad 0 \leq \rho \leq 1$ and $\gamma = \vartheta + \rho - \vartheta\rho$. Then*

(1) $(\nabla_m^{\vartheta,\rho}\eta)(\omega) = (\nabla_{m+1}^{-\rho(1-\vartheta)}\nabla_m^{\gamma}\eta)(\omega)$, $\omega \in \mathbb{N}_{m+1}$.
(2) $(\nabla_{m+1}^{-\gamma}\nabla_m^{\gamma}\eta)(\omega) = (\nabla_{m+1}^{-\vartheta}\nabla_m^{\vartheta,\rho}\eta)(\omega)$, $\omega \in \mathbb{N}_{m+1}$.
(3) $(\nabla_m^{\gamma}\nabla_m^{-\vartheta}\eta)(\omega) = (\nabla_m^{\rho(1-\vartheta)}\eta)(\omega)$, $\omega \in \mathbb{N}_{m+1}$.
(4) $(\nabla_m^{\vartheta,\rho}\nabla_m^{-\vartheta}\eta)(\omega) = (\nabla_{m+1}^{-\rho(1-\vartheta)}\nabla_m^{\rho(1-\vartheta)}\eta)(\omega)$, $\omega \in \mathbb{N}_{m+1}$.

Let us examine the initial value problem
$$(\nabla_m^{\vartheta,\rho}\eta)(\omega) = \xi(\omega,\eta(\omega)), \omega \in \mathbb{N}_{m+1}$$
$$[(\nabla_m^{-(1-\gamma)}\eta)(\omega)]_{\omega=m} = \eta(m) = \eta_0,$$
where $0 < \vartheta \leq 1$, $0 \leq \rho \leq 1$, and $\gamma = \vartheta + \rho - \vartheta\rho$, with $\xi : \mathbb{N}_m \times \mathbb{R} \to \mathbb{R}$.

Theorem 3.5 ([14])**.** *η satisfies the initial value problem* (1.1) *if, and only if, η is a solution of the Volterra summation equation*
$$\eta(\omega) = \eta_0\mathcal{H}_{\gamma-1}(\omega,\zeta(m)) + \sum_{s=m+1}^{\omega}\mathcal{H}_{\vartheta-1}(\omega,\zeta(s))\xi(s,\eta(s)), \quad \omega \in \mathbb{N}_m. \tag{3.1}$$

Let us consider the following assumptions:

(H1) Let $\xi(\omega,\eta(\omega))$ satisfies the Lipschitz condition
$$|\xi(\omega,\eta_1)-\xi(\omega,\eta_2)|\le A|\eta_1-\eta_2|,$$
where $0<A<1$ is a independent of ω.

(H2) If there exists a constant $M>0$ such that $|\xi(\omega,\eta(\omega))|\le M$ for $\omega\in\mathbb{N}_1$.

Theorem 3.6. *Let $\xi(\omega,\eta(\omega))$ be a nonlinear function which satisfies the assumption* (H1), (H2)*. Then the fractional difference equation* (1.1) *has at least one solution.*

Proof. Define the sequence $\{g_n(\cdot):n\in\mathbb{N}_0\}$, $g_0=\frac{(\omega-m+1)^{\overline{(\gamma-1)}}}{\Gamma(\gamma)}\eta(m)$, for $\omega\in\mathbb{N}_m$,
$$g_n(\omega)=\frac{(\omega-m+1)^{\overline{(\gamma-1)}}}{\Gamma(\gamma)}\eta(m)+\frac{1}{\Gamma(\vartheta)}\sum_{s=m+1}^{\omega}(\omega-\zeta(s))^{\overline{(\vartheta-1)}}\xi(s,g_{n-1}(s)),$$
$\omega\in\mathbb{N}_m,n\in\mathbb{N}_0$ Clearly, by induction, we have $|g_n-g_{n-1}|\le\frac{A^{n-1}M}{\Gamma(n\vartheta+1)}(\omega-m+1)^{\overline{(n\vartheta)}}$. Clearly, by induction, we have
$$|g_n(\omega)-g_{n-1}(\omega)|\le MA^{n-2}\frac{(\omega-m+1)^{\overline{(n-1)\vartheta}}}{\Gamma(n\vartheta+1)}$$
In fact, for $n=1$, by condition (H1) we can conclude that
$$\begin{aligned}|g_1(\omega)-g_0(\omega)|&=\frac{1}{\Gamma(\vartheta)}\sum_{s=m+1}^{\omega}(\omega-\zeta(s))^{\overline{(\vartheta-1)}}\xi(s,g_0(s))\\&\le\frac{1}{\Gamma(\vartheta)}\sum_{s=m+1}^{\omega}(\omega-\zeta(s))^{\overline{(\vartheta-1)}}M\\&\le\frac{M}{\Gamma(\vartheta+1)}(\omega-m+1)^{\overline{\vartheta}}.\end{aligned}$$
Without loss of generality, we assume that
$$|g_{n-1}(\omega)-g_{n-2}(\omega)|\le MA^{n-2}\frac{(\omega-m+1)^{\overline{(n-1)\vartheta}}}{\Gamma((n-1)\vartheta+1)}.$$
Then
$$\begin{aligned}|g_n(\omega)-g_{n-1}(\omega)|&\le\frac{A}{\Gamma(\vartheta)}\sum_{s=m+1}^{\omega}(\omega-\zeta(s))^{\overline{(\vartheta-1)}}MA^{n-2}\frac{(s-m+1)^{\overline{(n-1)\vartheta}}}{\Gamma((n-1)\vartheta+1)}\\&=\frac{MA^{n-1}}{\Gamma(n\vartheta+1)}\nabla_a^{-\vartheta}(\omega-m+1)^{\overline{n\vartheta}}\\&=MA^{n-1}\frac{(\omega-m+1)^{\overline{n\vartheta}}}{\Gamma(n\vartheta+1)}.\end{aligned}$$
Set
$$g(t)=\lim_{n\to\infty}(g_n(\omega)-g_0(\omega))+g_0(\omega)=\sum_{k=1}^{\infty}(g_k(\omega)-g_{k-1}(\omega))+g_0(\omega).$$

Since the series $\frac{M}{A}\sum_{k=1}^{\infty} A^k \frac{(\omega-m+1)^{\overline{n\vartheta}}}{\Gamma(n\vartheta+1)}$ is absolutely convergence for $0 < A < 1$, the existence of the solution for the fractional difference equation (1.1) is proved. The proof is complete. □

Let $\mathcal{Z}$ be the set of all real sequences $\eta = \{\eta(\omega)\}_{\omega=m}^{T}$ with $\|\eta\| = \sup_{\omega\in\mathbb{N}_m^T} |\eta(\omega)|$ is a Banach space. We define an operator $\mathcal{A} : \mathcal{Z} \to \mathcal{Z}$ as follows:

$$\mathcal{A}\eta(\omega) = \eta_0\mathcal{H}_{\gamma-1}(\omega, \zeta(m)) + \sum_{s=m+1}^{\omega} \mathcal{H}_{\vartheta-1}(\omega, \zeta(s))\xi(s, \eta(s)), \quad \omega \in \mathbb{N}_m. \tag{3.2}$$

The fixed points of $\mathcal{A}$ are identical to the solutions of problem (1.1).

Theorem 3.7. *Let $g : [m, T]_{\mathbb{N}_m} \to \mathbb{R}$ be a bounded function such that $|\xi(\omega, \eta)| \leq g(\omega)|\eta|$. Then* (1.1) *has at least one solution on $\mathcal{Z}$, provided that*

$$L^* \leq \frac{\Gamma(\vartheta+1)}{(T-m+1)^{\overline{\vartheta}}},$$

where $L^* = \sup_{\omega\in\mathbb{N}_m^T} g(\omega)$.

Proof. For a positive number M, we define the set $W = \{\eta : \|\eta-\eta_0 H_{\gamma-1}(\omega, \zeta(m))\| \leq M$, for $\omega \in \mathbb{N}_m^T$. We have to show that $\mathcal{A}$ maps W into itself. For $\eta \in W$, we have

$$\begin{aligned}|\mathcal{A}\eta(\omega) - \eta_0\mathcal{H}_{\gamma-1}(\omega, \zeta(m))| &\leq g(\omega) \sum_{s=m+1}^{\omega} \mathcal{H}_{\vartheta-1}(\omega, \zeta(s))|\eta(s)| \\ &\leq L^* \sup_{\omega\in\mathbb{N}_m^T} |\eta(\omega)| \sum_{s=m+1}^{\omega} \mathcal{H}_{\vartheta-1}(\omega, \zeta(s)) \\ &\leq L^*\|\eta\| \frac{(\omega-m+1)^{\overline{\vartheta}}}{\Gamma(\vartheta+1)} \\ &\leq L^*\|\eta\| \frac{(T-m+1)^{\overline{\vartheta}}}{\Gamma(\vartheta+1)} \leq M.\end{aligned}$$

We have $\|\mathcal{A}\eta\| \leq M$. It follows that $\mathcal{A}$ is self map. Therefore, according to Brouwer's fixed point theorem, $\mathcal{A}$ has at least one fixed point. □

Theorem 3.8. *For $K > 0$ and $\eta_1, \eta_2 \in \mathcal{Z}$, assume that $|\xi(\omega, \eta_1) - \xi(\omega, \eta_2)| \leq K|\eta_1 - \eta_2|$ for all $\omega \in [m, T]_{\mathbb{N}_m}$. Then* (1.1) *has a unique solution on $\mathcal{Z}$, provided that*

$$K \leq \frac{\Gamma(\vartheta+1)}{(T-m+1)^{\overline{\vartheta}}}. \tag{3.3}$$

Proof. Let $\eta_1, \eta_2 \in \mathcal{Z}$ and $\omega \in [m, T]_{\mathbb{N}_m}$. By assumption we have

$$\begin{aligned}|\mathcal{A}\eta_1(\omega) - \mathcal{A}\eta_2(\omega)| &\leq |\sum_{s=m+1}^{\omega} \mathcal{H}_{\vartheta-1}(\omega, \zeta(s))||\xi(\omega, \eta_1) - \xi(\omega, \eta_2)| \\ &\leq \frac{(\omega-m+1)^{\overline{\vartheta}}}{\Gamma(\vartheta+1)} K|\eta_1 - \eta_2|.\end{aligned}$$

Taking the supremum on both sides we have

$$\sup_{\omega\in\mathbb{N}_m^T} |\mathcal{A}\eta_1(\omega) - \mathcal{A}\eta_2(\omega)| \leq K\frac{(T-m+1)^{\overline{\vartheta}}}{\Gamma(\vartheta+1)}\|\eta_1 - \eta_2\|.$$

By using equation (3.3), we obtain $\|\mathcal{A}\eta_1 - \mathcal{A}\eta_2\| \le \|\eta_1 - \eta_2\|$, from which it follows that $\mathcal{A}$ is contraction mapping. Therefore, according to Banach's fixed point theorem $\mathcal{A}$ has a unique fixed point. □

Lemma 3.9. *Let $\eta : \mathbb{N}_m \to \mathbb{R}$, $0 \le \rho \le 1$, and $0 < \vartheta \le 1$. The $\vartheta^{th}-$order and $\rho^{th}-$type Hilfer nabla difference of η is defined by*

$$(\nabla_m^{\vartheta,\rho}\eta)(\omega) = (\nabla_{m+1}^{-\rho(1-\vartheta)}\nabla\nabla_m^{-(1-\rho)(1-\vartheta)}\eta)(\omega), \quad \omega \in \mathbb{N}_{m+1}.$$

Then

$$(\nabla_m^{\vartheta,\rho}\eta)(\omega) = (\nabla_m^{\vartheta}\eta)(\omega) - \mathcal{H}_{\rho(1-\vartheta)-1}(\omega,\zeta(m))(\nabla_m^{-(1-\gamma)}\eta)(m), \tag{3.4}$$

where $\gamma = \vartheta + \rho - \vartheta\rho$.

Proof. Evidently, $0 < \gamma \le 1$, Lemma 2.5 and equation (2.2) support this assertion. Consider

$$\begin{aligned}
&(\nabla_m^{\vartheta,\rho}\eta)(\omega)\\
&= (\nabla_{m+1}^{-\rho(1-\vartheta)}\nabla\nabla_m^{-(1-\rho)(1-\vartheta)}\eta)(\omega)\\
&= (\nabla\nabla_m^{-\rho(1-\vartheta)}\nabla_m^{-(1-\rho)(1-\vartheta)}\eta)(\omega) - \mathcal{H}_{\rho(1-\vartheta)-1}(\omega,\zeta(m))(\nabla_m^{-(1-\rho)(1-\vartheta)}\eta)(m)\\
&= (\nabla_m^{1-\rho(1-\vartheta)}\nabla_m^{-(1-\rho)(1-\vartheta)}\eta)(\omega) - \mathcal{H}_{\rho(1-\vartheta)-1}(\omega,\zeta(m))(\nabla_m^{-(1-\gamma)}\eta)(m)\\
&= (\nabla_m^{\vartheta}\eta)(\omega) - \mathcal{H}_{\rho(1-\vartheta)-1}(\omega,\zeta(m))(\nabla_m^{-(1-\gamma)}\eta)(m).
\end{aligned}$$

□

Lemma 3.10. *For each discrete time $\omega \in \mathbb{N}_{m+1}$, where $0 < \vartheta \le 1$, $0 \le \rho \le 1$ and $\gamma = \vartheta + \rho - \vartheta\rho$, the inequality involving Hilfer difference holds*

$$(\nabla_m^{\vartheta,\rho}\frac{\eta^2}{2})(\omega) \le \eta(\omega)(\nabla_m^{\vartheta,\rho}\eta)(\omega),$$

provided $\eta(\omega) \le \frac{1}{2}\eta(m)$, $\omega \in \mathbb{N}_{m+1}$.

Proof. Note that

$$\begin{aligned}
&\eta(\omega)(\nabla_m^{\vartheta,\rho}\eta)(\omega) - (\nabla_m^{\vartheta,\rho}\frac{\eta^2}{2})(\omega)\\
&= \eta(\omega)(\nabla_m^{\vartheta}\eta)(\omega) - \eta(\omega)\mathcal{H}_{\rho(1-\vartheta)-1}(\omega,\zeta(m))\eta(m)\\
&\quad - (\nabla_m^{\vartheta}\frac{\eta^2}{2})(\omega) + \mathcal{H}_{\rho(1-\vartheta)-1}(\omega,\zeta(m))\frac{\eta^2}{2}(m).
\end{aligned}$$

Using [27, Lemma 3.9], $\eta(\omega)(\nabla_m^{\vartheta,\rho}\eta)(\omega) - (\nabla_m^{\vartheta,\rho}\frac{\eta^2}{2})(\omega) \ge 0$ when $\eta(\omega) \le \frac{1}{2}\eta(m)$, $\omega \in \mathbb{N}_{m+1}$. □

4. Stability theorems

The stability analysis of fractional differential equations has been explored extensively. In their study, the authors utilized Laplace transforms to derive a Mittag-Leffler solution. However, it is crucial to note that continuity in solutions does not persist in fractional difference equations, thus hindering the use of traditional mathematical analysis tools to assess the asymptotic behavior of these equations. Therefore, alternative methodologies are required to investigate stability. To begin, let's revisit some stability definitions pertinent to discrete systems in nonlinear dynamics.

Definition 4.1 ([27])**.** Let for $\eta^* = (\eta_1(\omega), \cdots, \eta_N(\omega))^T$, denote a vector function at time $\omega \in \mathbb{N}_{m+1}$, consider the equilibrium point $\eta = 0$.

$$\nabla_{\omega_0}^{\vartheta,\rho}\eta^*(\omega) = \xi(\omega, \eta^*(\omega)),\ \eta^*(\omega_0) = C, \quad 0 < \vartheta \leq 1,\ 0 \leq \rho \leq 1,\ \omega \in \mathbb{N}_{\omega_0+1}.$$

The equilibrium point is said to be stable if for all $\epsilon > 0$ there exits a $\delta = \delta(\omega_0, \epsilon) > 0$ such that if $\|\eta^*(\omega_0)\| < \delta$ then $\|\eta^*(\omega)\| < \epsilon$, for all $\omega > \omega_0$, $\omega \in \mathbb{N}_{\omega_0}$.

Definition 4.2 ([27])**.** An equilibrium point is asymptotically stable if for all $\epsilon > 0$ there exits a, $\delta = \delta(\omega_0) > 0$ such that $\|\eta^*(\omega_0)\| < \delta$ implies $\lim_{\omega\to\infty} \eta^* = 0$, $\omega \in \mathbb{N}_{\omega_0}$.

Using the discrete fractional Lyapunov direct technique, we examine the stability of equation (1.1). For simplicity, we assume $\omega_0 = m$ in the remainder of the work.

Lemma 4.3. *Let $\eta(m)$, $\quad \eta(m+1) > 0$, $\eta(\omega)$ is a solution of*

$$(\nabla_m^{\vartheta,\rho}\eta)(\omega) = \lambda\eta(\omega), \omega \in \mathbb{N}_{m+1}, \quad 0 < \vartheta \leq 1, 0 \leq \rho \leq 1, \tag{4.1}$$

and $0 \leq \lambda < 1$ then $\eta(\omega) > 0$ for all $\omega \in \mathbb{N}_{m+1}$.

Proof. Using Theorem 3.5, we have

$$\eta(\omega) = \eta_0\mathcal{H}_{\gamma-1}(\omega, \zeta(m)) + \lambda\sum_{s=m+1}^{\omega}\mathcal{H}_{\vartheta-1}(\omega, \zeta(s))\eta(s)$$

$$\eta(\omega) = \eta_0\mathcal{H}_{\gamma-1}(\omega, \zeta(m)) + \lambda\sum_{s=m+1}^{\omega-1}\mathcal{H}_{\vartheta-1}(\omega, \zeta(s))\eta(s) + \lambda\mathcal{H}_{\vartheta-1}(\omega, \zeta(\omega))\eta(\omega)$$

$$\eta(\omega) = \eta_0\mathcal{H}_{\gamma-1}(\omega, \zeta(m)) + \lambda\sum_{s=m+1}^{\omega-1}\mathcal{H}_{\vartheta-1}(\omega, \zeta(s))\eta(s) + \lambda\eta(\omega)$$

$$(1-\lambda)\eta(\omega) = \eta_0\mathcal{H}_{\gamma-1}(\omega, \zeta(m)) + \lambda\sum_{s=m+1}^{\omega-1}\mathcal{H}_{\vartheta-1}(\omega, \zeta(s))\eta(s)$$

$$\eta(\omega) = \frac{\eta_0}{1-\lambda}\mathcal{H}_{\gamma-1}(\omega, \zeta(m)) + \frac{\lambda}{1-\lambda}\sum_{s=m+1}^{\omega-1}\mathcal{H}_{\vartheta-1}(\omega, \zeta(s))\eta(s)$$

$$\eta(\omega) = \frac{\eta_0}{1-\lambda}\frac{\Gamma(\omega-m+\gamma)}{\Gamma(\omega-m+1)\Gamma(\gamma)} + \frac{\lambda}{1-\lambda}\sum_{s=m+1}^{\omega-1}\frac{\Gamma(\omega-s+\vartheta)}{\Gamma(\omega-s+1)\Gamma(\vartheta)}\eta(s).$$

Considering the conditions $0 \leq \lambda < 1$, $\frac{\Gamma(\omega-m+\gamma)}{\Gamma(\omega-m+1)\Gamma(\gamma)} > 0$ and $\frac{\Gamma(\omega-s+\vartheta)}{\Gamma(\omega-s+1)\Gamma(\vartheta)} > 0$ for $s = m+1, \ldots, \omega-1$, we can conclude that if $\eta(m+1) > 0$, then $\eta(m+2) > 0$. Subsequently, this leads to the inference that $\eta(\omega) > 0$ for all $\omega \in \mathbb{N}_{m+1}$. □

Lemma 4.4. *Let $\eta(\omega)$ and $\nu(\omega)$ satisfy*

$$(\nabla_m^{\vartheta,\rho}\eta)(\omega) = \lambda\eta(\omega), \quad \omega \in \mathbb{N}_{m+1},\ 0 < \vartheta \leq 1,\ 0 \leq \rho \leq 1, \tag{4.2}$$

and the inequality

$$(\nabla_m^{\vartheta,\rho}\nu)(\omega) \leq \lambda\nu(\omega), \quad \omega \in \mathbb{N}_{m+1}, \tag{4.3}$$

respectively. If $0 \leq \lambda < 1$ and $\eta(m+1) = \nu(m+1) > 0$, then $\nu(\omega) \leq \eta(\omega)$ for $\omega \in \mathbb{N}_{m+1}$.

Proof. Using Theorem 3.5, we have

$$\eta(\omega) = \frac{\eta_0}{1-\lambda}\mathcal{H}_{\gamma-1}(\omega,\zeta(m)) + \frac{\lambda}{1-\lambda}\sum_{s=m+1}^{\omega-1}\mathcal{H}_{\vartheta-1}(\omega,\zeta(s))\eta(s)$$

$$\eta(\omega) = \frac{\eta_0}{1-\lambda}\frac{\Gamma(\omega-m+\gamma)}{\Gamma(\omega-m+1)\Gamma(\gamma)} + \frac{\lambda}{1-\lambda}\sum_{s=m+1}^{\omega-1}\frac{\Gamma(\omega-s+\vartheta)}{\Gamma(\omega-s+1)\Gamma(\vartheta)}\eta(s),$$

and

$$\nu(\omega) \le \frac{\eta_0}{1-\lambda}\mathcal{H}_{\gamma-1}(\omega,\zeta(m)) + \frac{\lambda}{1-\lambda}\sum_{s=m+1}^{\omega-1}\mathcal{H}_{\vartheta-1}(\omega,\zeta(s))\nu(s)$$

$$\nu(\omega) \le \frac{\eta_0}{1-\lambda}\frac{\Gamma(\omega-m+\gamma)}{\Gamma(\omega-m+1)\Gamma(\gamma)} + \frac{\lambda}{1-\lambda}\sum_{s=m+1}^{\omega-1}\frac{\Gamma(\omega-s+\vartheta)}{\Gamma(\omega-s+1)\Gamma(\vartheta)}\nu(s).$$

If we set $\omega = m+2$ we can establish that $\nu(m+2) \le \eta(m+2)$. Assuming the inequality holds true for $\nu(m+k) \le \eta(m+k)$ for $n = m+k$, $k > 2$, then through the application of the principle of induction, we can readily demonstrate that the inequality remains valid for $n = m+k+2$, where $k > 2$. □

Lemma 4.5 (Discrete comparison principle). *For $0 < \vartheta \le 1, 0 \le \rho \le 1$ and $\gamma = \vartheta + \rho - \vartheta\rho$, if $\nabla_m^{\vartheta,\rho} y(\omega) \le \nabla_m^{\vartheta,\rho} x(\omega)$, for all $\omega \in \mathbb{N}_{m+1}$ and $x(m) = y(m)$, then $y(\omega) \le x(\omega)$.*

Proof. Let $F(\omega) = x(\omega) - y(\omega)$. Given

$$\nabla_m^{\vartheta,\rho} x(\omega) \ge \nabla_m^{\vartheta,\rho} y(\omega),$$
$$\nabla_m^{\vartheta,\rho} F(\omega) \ge 0.$$

We apply $\nabla_{m+1}^{-\vartheta}$ on both side to obtain

$$\nabla_{m+1}^{-\vartheta}\nabla_m^{\vartheta,\rho} F(\omega) \ge 0.$$

Using Proposition 3.4(2),

$$(\nabla_{m+1}^{-\gamma}\nabla_m^{\gamma} F)(\omega) \ge 0,$$
$$\nabla_{m+1}^{-\gamma}\nabla(\nabla_m^{-(1-\gamma)} F)(\omega) \ge 0.$$

By using Lemma 3.9,

$$(\nabla\nabla_m^{-\gamma}\nabla_m^{-(1-\gamma)} F)(\omega) - \mathcal{H}_{\gamma-1}(\omega,\zeta(m))(\nabla_m^{(1-\gamma)} F)(m) \ge 0,$$
$$(\nabla\nabla_m^{-\gamma}\nabla_m^{-(1-\gamma)} F)(\omega) \ge 0, F(\omega) \ge 0;$$

therefore, $x(\omega) \ge y(\omega)$. □

Theorem 4.6. *Let $\eta = 0$ be an equilibrium point of (1.1). If there exits a positive definite and decrescent scalar function $V(\omega, \eta(\omega))$ and discrete class-$\mathcal{K}$ functions γ_1, γ_2 and γ_3 such that*

$$\gamma_1(\|\eta(\omega)\|) \le V(\omega,\eta(\omega)) \le \gamma_2(\|\eta(\omega)\|), \quad \omega \in \mathbb{N}_{m+1}, \tag{4.4}$$
$$\nabla_m^{\vartheta,\rho} V(\omega,\eta(\omega)) \le -\gamma_3(\|\eta(\omega)\|), \quad \omega \in \mathbb{N}_{m+2}, \tag{4.5}$$

then the equilibrium point is asymptotically stable.

Proof. From the inequalities we obtain

$$\|\eta(\omega)\| \geq \gamma_2^{-1}(V(\omega,\eta(\omega))),$$
$$\nabla_m^{\vartheta,\rho} V(\omega,\eta(\omega)) \leq -\gamma_3(\gamma_2^{-1}(V(\omega,\eta(\omega))), \quad \omega \in \mathbb{N}_{m+2},$$

where γ_2^{-1} is the inverse of the discrete class-$\mathcal{K}$ function γ_2. We construct a fractional difference equation

$$\nabla_m^{\vartheta,\rho} U(\omega) = -\gamma_3(\gamma_2^{-1}(U(\omega))), \quad U(m+1) = V(m+1,\eta(m+1)) > 0.$$

$V(\omega,\eta(\omega))$ is bounded by the solution $U(\omega)$, according to Lemma 4.3, $V(\omega,\eta(\omega))$ is a positive scalar function. We can derive $\lim_{\omega\to\infty} V(\omega,\eta(\omega)) = 0$ according to Lemma 4.3 and 4.4. Since γ_1 is discrete class function, we derive that $\lim_{\omega\to\infty}\eta(\omega) = 0$. □

Theorem 4.7. *If* $\eta(\omega)\xi(\omega,\eta(\omega)) < 0$ *and* $\eta(\omega) \leq \frac{\eta(m)}{2}$*. Then discrete equation* (1.1) *is asymptotically stable.*

Proof. Let us consider the Lyapunov function $V = \frac{\eta^2(\omega)}{2}$. Using Lemma 3.10 we can estimate the fractional derivative of V,

$$\nabla_m^{\vartheta,\rho} V \leq \eta(\omega)\nabla_m^{\vartheta,\rho}\eta(\omega) = \eta(\omega)\xi(\omega,\eta(\omega)) < 0, \quad \omega \in \mathbb{N}_{m+1}.$$

where $\nabla_m^{\vartheta,\rho} V$ is negative definite. Hence by Theorem 4.6 the discrete system (1.1) is asymptotically stable. □

5. Examples

We examine a few uses for the asymptotic stability in this section.

Example 5.1. The discrete fractional equations for $0 \leq \rho \leq 1$, $0 < \vartheta \leq 1$, and $\gamma = \vartheta + \rho - \vartheta\rho$ are examined on $\mathbb{N}_{m+1}$.

$$\begin{aligned}
\nabla_m^{\vartheta,\rho} x(\omega) &= \sigma(y(\omega) - x(\omega)), \quad x(m+1) = 0.1,\\
\nabla_m^{\vartheta,\rho} y(\omega) &= rx(\omega) - y(\omega) - x(\omega)z(\omega), \quad y(m+1) = 0.2,\\
\nabla_m^{\vartheta,\rho} z(\omega) &= x(\omega)y(\omega) - bz(\omega), \quad z(m+1) = 0.3, \quad \omega \in \mathbb{N}_{m+2},
\end{aligned} \tag{5.1}$$

where σ, r and b are three positive constants. Let us consider the Lyapunov candidate function which is positive definite

$$V(x(\omega),y(\omega),z(\omega)) = \frac{x^2(\omega) + \sigma y^2(\omega) + \sigma z^2(\omega)}{2},$$

$$\begin{aligned}
&\nabla_m^{\vartheta,\rho} V(x(\omega),y(\omega),z(\omega))\\
&\leq x(\omega)\nabla_m^{\vartheta,\rho} x(\omega) + \sigma y(\omega)\nabla_m^{\vartheta,\rho} y(\omega) + \sigma z(\omega)\nabla_m^{\vartheta,\rho} z(\omega)\\
&= -\frac{\sigma(1+r)}{2}(x(\omega) - y(\omega))^2 - \frac{\sigma(1-r)}{2}x^2(\omega) - \frac{\sigma(1-r)}{2}y^2(\omega) - b\sigma z^2(\omega),
\end{aligned}$$

for $\omega \in \mathbb{N}_{m+2}$. By using Theorem 4.6, this system is asymptotically stable if $0 < r < 1$, and r is independent of the fractional order ϑ.

Case I: By using Lemma 3.9, we have

$$\frac{1}{\Gamma(-\vartheta)}\sum_{j=1}^{n}\frac{\Gamma(n-j-\vartheta)}{\Gamma(n-j+1)}x_j = \sigma(y_n - x_n) + \frac{\Gamma(n+\rho(1-\vartheta))}{\Gamma(n+1)\Gamma(\rho(1-\vartheta))}x_1, \quad x_1 = 0.1,$$

$$\frac{1}{\Gamma(-\vartheta)}\sum_{j=1}^{n}\frac{\Gamma(n-j-\vartheta)}{\Gamma(n-j+1)}y_j = rx_n - y_n - x_nz_n + \frac{\Gamma(n+\rho(1-\vartheta))}{\Gamma(n+1)\Gamma(\rho(1-\vartheta))}y_1, \quad y_1 = 0.2,$$

$$\frac{1}{\Gamma(-\vartheta)}\sum_{j=1}^{n}\frac{\Gamma(n-j-\vartheta)}{\Gamma(n-j+1)}z_j = x_ny_n - bz_n + \frac{\Gamma(n+\rho(1-\vartheta))}{\Gamma(n+1)\Gamma(\rho(1-\vartheta))}z_1, \quad z_1 = 0.3,$$

where $n \geq 2$, $x_j = x(m+j)$, $y_j = y(m+j)$ and $z_j = z(m+j)$.

Case II: From the equivalent sum equation from Theorem 3.5, we obtain

$$x_{n+2} = \frac{\Gamma(n+\gamma+2)}{\Gamma(n+3)\Gamma\gamma}x_1 + \sigma\sum_{j=0}^{n}\frac{\Gamma(n-j+\vartheta)}{\Gamma(n-j+1)\Gamma\vartheta}(y_{j+2}-x_{j+2}), \quad x_1 = 0.1,$$

$$y_{n+2} = \frac{\Gamma(n+\gamma+2)}{\Gamma(n+3)\Gamma\gamma}y_1 + \sum_{j=0}^{n}\frac{\Gamma(n-j+\vartheta)}{\Gamma(n-j+1)\Gamma\vartheta}(rx_{j+2}-y_{j+2}-x_{j+2}z_{j+2}), \quad y_1 = 0.2,$$

$$z_{n+2} = \frac{\Gamma(n+\gamma+2)}{\Gamma(n+3)\Gamma\gamma}z_1 + \sum_{j=0}^{n}\frac{\Gamma(n-j+\vartheta)}{\Gamma(n-j+1)\Gamma\vartheta}(x_{j+2}y_{j+2}-bz_{j+2}), \quad z_1 = 0.3,$$

where $n \geq 1$.

Case I and Case II are equivalent, they lead to the same conclusion. In Case II, x_{n+1}, y_{n+1} and z_{n+1} form an implicit algebra system for $1 \leq n$. For $\sigma = 1, r = 0.5$ and $b = 1$, we show the case of $\vartheta = 0.95$, $\rho = 0.90$, $\gamma = 0.99$ and $\vartheta = 0.92$, $\rho = 0.82$, $\gamma = 0.98$, and $\vartheta = 0.9$, $\rho = 0.8$, $\gamma = 0.98$ and $\vartheta = 0.88$, $\rho = 0.78$, $\gamma = 0.97$. We note that the numerical solutions tends to the origin when $\omega \to \infty$ in figures (A), (B), (C) and (D) respectively.

Example 5.2. The discrete fractional equations for $0 \leq \rho \leq 1$, $0 < \vartheta \leq 1$, and $\gamma = \vartheta + \rho - \vartheta\rho$ are examined. They are defined on $\mathbb{N}_{m+1}$.

$$\begin{aligned}\nabla_m^{\vartheta,\rho}x(\omega) &= -x(\omega) + y^3(\omega), \quad x(m+1) = 0.1,\\ \nabla_m^{\vartheta,\rho}y(\omega) &= -x(\omega) - y(\omega), \quad y(m+1) = 0.2.\end{aligned} \tag{5.2}$$

Let us consider the Lyapunov candidate function which is positive definite.

$$\begin{aligned}V(x(\omega), y(\omega)) &= \frac{1}{2}x^2(\omega) + \frac{1}{4}y^4(\omega),\\ \nabla_m^{\vartheta,\rho}V(x(\omega), y(\omega)) &= \frac{1}{2}\nabla_m^{\vartheta,\rho}x^2(\omega) + \frac{1}{4}\nabla_m^{\vartheta,\rho}y^4(\omega)\\ &\leq x(\omega)\nabla_m^{\vartheta,\rho}x(\omega) + y^3(\omega)\nabla_m^{\vartheta,\rho}y(\omega)\\ &\leq x(\omega)(-x(\omega) + y^3(\omega)) + y^3(\omega)(-x(\omega) - y(\omega))\\ &< -x^2(\omega) - y^4(\omega) < 0.\end{aligned}$$

By using Theorem 4.6 this system is asymptotically stable.

Here we used the Newton-iteration method for the solutions. The asymptotic stability of the system is shown in Figure 1(e) where $\vartheta = 0.95$, $\rho = 0.75$, $\gamma = 0.99$.

$$\begin{aligned}x_{n+2} &= \frac{\Gamma(n+\gamma+2)}{\Gamma(n+3)\Gamma\gamma}x_1 + \sum_{j=0}^{n}\frac{\Gamma(n-j+\vartheta)}{\Gamma(n-j+1)\Gamma\vartheta}(-x_{j+2}+y_{j+2}^3), \quad x_1 = 0.1,\\ y_{n+1} &= \frac{\Gamma(n+\gamma+2)}{\Gamma(n+3)\Gamma\gamma}y_1 + \sum_{j=0}^{n}\frac{\Gamma(n-j+\vartheta)}{\Gamma(n-j+1)\Gamma\vartheta}(-x_{j+2}-y_{j+2}), \quad y_1 = 0.2,\end{aligned} \tag{5.3}$$

where $n \geq 1$.

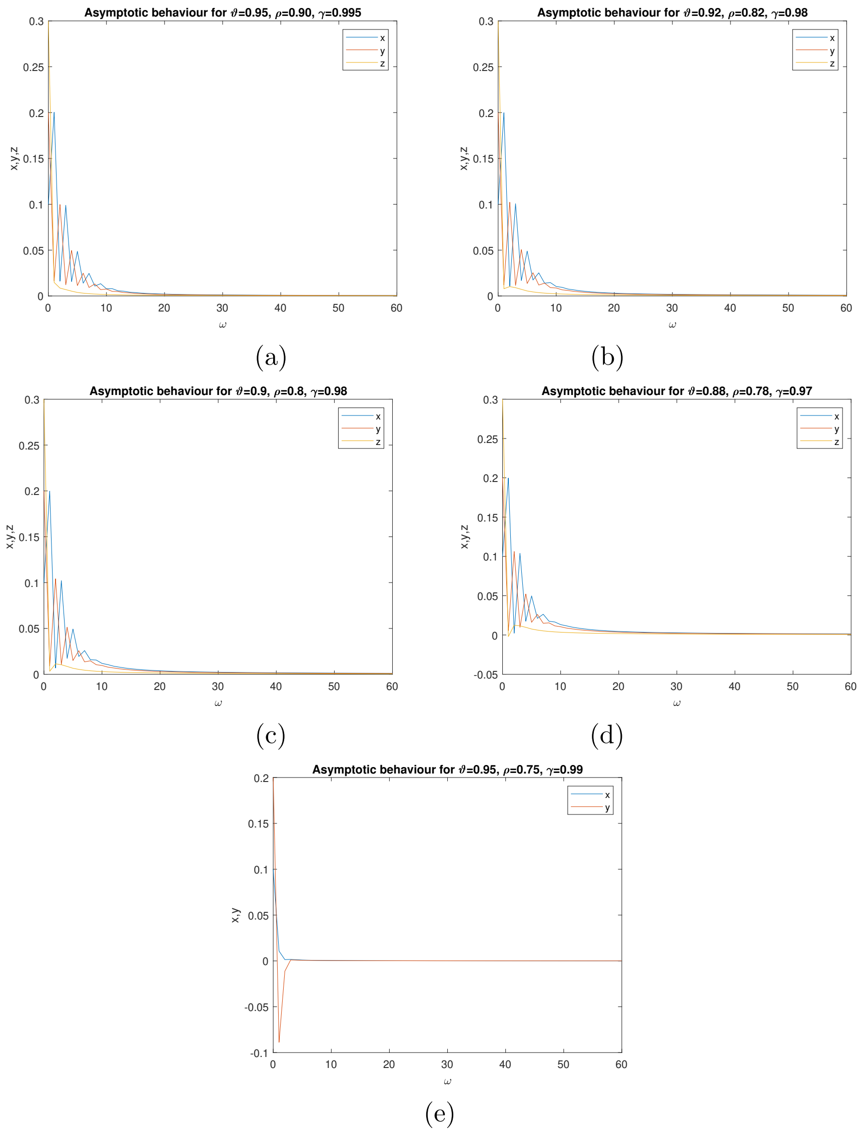

FIGURE 1. IAsymptotic behavior of solutions for various values of parameters ϑ, ρ, and γ.

Conclusion. We examined the existence and uniqueness theorem, asymptotic stability of fractional nonlinear difference equations. Since the exact solution is not known, we are unable to determine the asymptotic stability through the traditional approach of utilizing solution properties. This study utilized asymptotic analysis of a nonlinear fractional difference equation. We established the positivity of the solution and formulated a fractional scalar difference equation. Additionally, we provided a proof of asymptotic stability using the Lyapunov direct method. Two numerical examples with simulation results are discussed, employing both the well-known Lyapunov direct method and the Newton iteration method.

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