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INITIAL TEMPERATURE PROFILE RECOVERY UNDER A MIXED BOUNDARY CONDITION

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ABSTRACT. The challenge of finding the initial temperature distribution (profile) has been addressed for different boundary conditions. Previous studies studied this problem under Dirichlet [2, 6], Neumann [12], and periodic [11]. This article focuses on the problem with mixed boundary conditions. We consider a one-dimensional body where temperatures are measured at a specific location x_0 and at a finite number of increasing future time points in a bounded interval.

1. Introduction

We consider the initial-boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0, \quad u_x(\pi,t) = 0, \quad u(x,0) = f(x), \tag{1.1}$$

This equation describes the behavior of the temperature distribution, denoted by $u(x,t)$, within a thin, uniform, one-dimensional rod of length π under the mixed boundary condition. These initial boundary conditions are also called as Dirichlet-Neumann boundary conditions.

For a known integrable function f over the interval $[0,\pi]$, it is well known that problem (1.1) has a solution with the Fourier sine series representation

$$u(x,t) = \sum_{k=1}^{\infty} e^{-(\frac{2k-1}{2})^2 t} \hat{f}_k \sin\Big(\frac{2k-1}{2}x\Big). \tag{1.2}$$

where $\hat{f}_k = \frac{2}{\pi}\int_0^\pi f(x)\sin\big(\frac{2k-1}{2}x\big)dx$. In particular, note that

$$f(x) = u(x,0) = \sum_{k=1}^{\infty} \hat{f}_k \sin\Big(\frac{2k-1}{2}x\Big).$$

But, what if we do not know the initial temperature profile $f(x)$? Suppose we only have temperature measurements $(u(x,t))$ at a specific location x_0 on the rod and at a finite future times. Can we still recover f with a certain desired rate of accuracy? This type of problem is highly ill-posed without further assumptions

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on f, x_0 and future time selections. This type of recovering the initial profile from discrete sample has been widely studied under the context of fractional Laplacian or under the different boundary conditions or other contexts in recent times (see [1, 2, 6, 8, 9, 10, 11, 12, 13]) and has many applications in many areas such as mathematical biology or physics (see [3, 7, 14, 15]).

To avoid ill-posedness and to recover an initial temperature profile under the Dirichlet boundary conditions, DeVore and Zuazua [6] have assumed that the initial data f lies in the following closed subset of $L^2([0,\pi])$:

$$\mathcal{F}_r = \Big\{ f \in L^2([0,\pi]) : \sum_{j=1}^{2r} j^{2r} |\hat{f}_j|^2 \le 1 \Big\} \tag{1.3}$$

where r is a positive real number. Furthermore, to avoid ill-posedness appearing due to vanishing temperature of eigenfunctions, they have selected x_0 to avoid the nodal sets of eigenfunctions $\sin kx, k = 1,2,3,\dots$ at x_0 and defined exponentially growing future times $t_k = (2\sqrt{2})^{k-1} t_1$, $k = 1,2,3,\dots$ with an arbitaly chosen $t_1 > 0$.

Later, Aryal and Karki [2] improved DeVore and Zuazua's result by selecting linearly growing finite future times.

$$t_k = (n + k - 1) t_1 \tag{1.4}$$

for $k = 2,3,\dots,n$ lie within a bounded interval $[0,T]$ by slightly modifying on the choice of f from the closed subset $\mathcal{F}_r$ of $L^2([0,\pi])$ to a smaller subset

$$\mathcal{B}_r = \Big\{ f \in L^2([0,\pi]) : \sum_{k=1}^{\infty} (k+2)^{2r} |\hat{f}_k|^2 \le 1 \Big\}, \tag{1.5}$$

Their time selection is far more practical in real-life applications than DeVore and Zuazua's exponentially growing time selection.

For the Neumann boundary conditions, the author (with Karki and Shawn) [12] used Fourier cosine represention and a modified $L^2([0,\pi])$ closed subspace to recover the initial profile.

For the periodic boundary conditions, the author (with Karki and Allison) [11] adopted complex analytic method by defining the L^2 closed subspace

$$\mathcal{C}_r = \Big\{ f \in L^2([0,\pi]) : \sum_{k=-\infty}^{\infty} (|k| + e)^{2r} |\hat{f}_k|^2 \le 1 \Big\}. \tag{1.6}$$

In particular, the complex analytic approach in [11] turns out to be a unified solution approach for Dirichlet and Neumann boundary conditions.

In this article, we establish a result corresponding to the initial-boundary value problem described in (1.1). For a future linear time selection, we use the time selection (1.4). We also assume that f is in $\mathcal{B}_r$, the closed subset of $L^2([0,\pi])$. To avoid nodal sets of the eigenfunctions, $\sin\big(\frac{2k-1}{2}x\big)$, $k = 1,2,3,\dots$, we need to choose an x_0 on the one-dimensional rod with $\sin\big(\frac{2k-1}{2}x_0\big) \ne 0$ for all $k = 1,2,3,\dots$. From Lemma 2.1 in [2], we know that there is an x_0 such that $|\sin kx_0| \ge d$ for some $d > 0$ for all $k = 1,2,3,\dots$. It is possible to select $0 \le x_0 \le \pi/2$. Now, define $x_0' = 2x_0$, where x_0 is the point in [2, Lemma 2.1]. Then $\sin\big(\frac{2k-1}{2}x_0'\big) = \sin\big((2k-1)x_0\big) \ne 0$ for all $k = 1,2,3,\dots$. Therefore, for eigenfunctions, $\sin\big(\frac{2k-1}{2}x\big)$, $k = 1,2,3,\dots$, we can choose x_0 as x_0'. Furthermore, we know that

$$\big|\sin\big(\frac{2k-1}{2}x_0\big)\big| \geq d \quad \text{for some } d > 0 \tag{1.7}$$

Finally, for any forward time sequence t_k as in (1.4), the corresponding discrete temperature measurements $u(x_0, t_k)$, $k = 1, 2, \dots$ are sufficient to determine the initial profile f uniquely. To see this, consider a holomorphic function

$$F(z) = \sum_{j=1}^{\infty} z^{(2j-1)^2} \hat{f}_j \sin\Big(\frac{2j-1}{2}x\Big),$$

where $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Choose $z_k = e^{-t_k/4} \in \mathbb{D}$ where $k = 1, 2, 3, \dots$. Since z_k converges in $\mathbb{D}$, we can uniquely determine F and hence $\hat{f}_j$. Therefore, $u(x_0, t_j)$ uniquely determine f.

2. Optimal approximation error

In this section, first, we briefly recall the theory of manifold width as discussed in [4, 5] to develop a measurement algorithm and recall a lower bound for the optimal error of approximation to an initial profile. Then, it is natural to ask whether there is an upper bound with a certain desired accuracy. In our main result, we tackle the upper bound problem.

2.1. Lower bound on optimal error.

From [1, 2, 6, 13], we briefly recall a measurement algorithm and then discuss a lower bound on the optimal error of the approximation. For reader's reference, we like to briefly discuss the approach here. The development measurement algorithm uses the theory of manifold width in [4, 5] and is indeed an encoder coupled with a decoder. An encoder is a continuous function that maps each element of a compact subspace $\mathcal{B}$ of $L^2([0,\pi])$ into a point in $\mathbb{R}^n$, and a decoder is a continuous function that maps each point $y \in \mathbb{R}^n$ into an element of $L^2([0,\pi])$. In our setting, an encoder is a continuous function e_n mapping $f \in \mathcal{B}$ into $e_n(f) = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$ where $u_k = u(x_0, t_k)$, $k = 1, 2, \dots, n$, is n temperature measurements. On the other hand, a decoder is a continuous function M_n mapping $(u_1, u_2, \dots, u_n) \in \mathbb{R}^n$ into an approximation $\bar{f}_n$ of f. The optimal error of approximation to f is defined as

$$\delta_{e_n, M_n}(\mathcal{B}_r, L^2) = \sup_{f \in \mathcal{B}_r} \|f - \bar{f}_n\|_{L^2} \tag{2.1}$$

where $\bar{f}_n = M_n(e_n(f))$.

In [2] we obtain a lower bound for this optimal error (also see [6]) as below.

Theorem 2.1. *For the measurement algorithm defined with an encoder $e_n : f \mapsto (u_1, u_2, \dots, u_n)$ and a continuous decoder $M_n : (u_1, u_2, \dots, u_n) \mapsto \bar{f}_n$ as described above, we have*

$$\delta_{e_n, M_n}(\mathcal{B}_r, L^2) \geq C n^{-r} \tag{2.2}$$

where C is a constant depending on r only.

2.2. **Main theorem.** Our main goal is to recover an initial profile f with a desired rate of accuracy order n^{-r} once we observe n temperature measurements $u(x_0, t_j),\ j = 1, 2, \dots, n$. More precisely, we have the following theorem.

Theorem 2.2. *Fix x_0 in $[0, \pi]$ as described in (1.7). Given $n \in \mathbb{N}$ and a bounded interval $[0, T]$, we choose $t_1 \in (0, T]$ such that the next $n - 1$ future times $t_j := (n + j - 1)t_1,\ j = 2, \dots n$ are also in the interval $[0, T]$. Let $f \in \mathcal{B}_r$. Then for the known n temperature measurements $u(x_0, t_j),\ j = 1, 2, \dots, n$, there exists $\bar{f}_n$, an approximation to f, in $L^2([0, \pi])$ such that*

$$\|f - \bar{f}_n\|_{L^2} \le Cn^{-r}, \tag{2.3}$$

where C is a constant that depends on r and t_1.

Setting $c_k = \hat{f}_k \sin\big(\frac{2k-1}{2}x_0\big)$ from (1.2), we first estimate c_k and construct an approximation of c_k. Then we give the proof of the theorem at the end of this section.

From [2, Lemma 2.3], we have the following Lemma with the same proof.

Lemma 2.3. *The coefficients $c_j = \hat{f}_j \sin\big(\frac{2k-1}{2}x_0\big)$ are bounded. More precisely,*

$$|c_j| \le (j + 2)^{-r}, \quad j = 1, 2, \dots.$$

Now, we define an approximation $\bar{c}_k$ of c_k. Later, we will use $\bar{c}_k$ when constructing an approximation $\bar{f}_n$ of f. Note that

$$\begin{aligned} u(x_0, t_k) &= \sum_{j=1}^{\infty} c_j e^{-(\frac{2j-1}{2})^2 t} \\ &= \sum_{j=1}^{k-1} c_j e^{-(\frac{2j-1}{2})^2 t} + c_k e^{-(\frac{2k-1}{2})^2 t} + \sum_{j=k+1}^{\infty} c_j e^{-(\frac{2j-1}{2})^2 t}, \quad k = 1, 2, 3, \dots, n. \end{aligned}$$

From this, we obtain

$$\begin{aligned} c_k &= u(x_0, t_k) e^{(\frac{2k-1}{2})^2 t} - \sum_{j=1}^{k-1} c_j e^{[(\frac{2k-1}{2})^2 - (\frac{2j-1}{2})^2] t_k} - \sum_{j=k+1}^{\infty} c_j e^{[(\frac{2k-1}{2})^2 - (\frac{2j-1}{2})^2] t_k} \\ &= u(x_0, t_k) e^{(\frac{2k-1}{2})^2 t} - \sum_{j=1}^{k-1} c_j e^{(k+j-1)(k-j) t_k} - \sum_{j=k+1}^{\infty} c_j e^{(k+j-1)(k-j) t_k} \end{aligned}$$

Now we define an approximation $\bar{c}_k$ to c_k as

$$\begin{aligned} \bar{c}_k := u(x_0, t_k) e^{(\frac{2k-1}{2})^2 t} &- \sum_{j=1}^{k-1} (j + 2)^{-r} e^{(k+j-1)(k-j) t_k} \\ &- \sum_{j=k+1}^{n} (j + 2)^{-r} e^{(k+j-1)(k-j) t_k} \end{aligned} \tag{2.4}$$

To calculate an error bound between c_k and $\bar{c}_k$, we will need a couple of estimates.

Lemma 2.4. *Let $p^* = T[(\frac{T}{2t_1} + \frac{3}{2})^2 - 2]$ and $r \ge p^*$. Then, for the n future times $t_k, k = 1, 2, \dots, n$ as in Theorem 2.2 and $j = 1, \dots, k - 1$, we obtain*

$$(k^2 - j^2 + k + j) t_k \le r.$$

Proof. Note that $t_k = (n+k-1)t_1 \le T$ for $k = 1, \dots, n$. Therefore,

$$t_n = (2n-1)t_1 \le T \Rightarrow n+1 \le \Big(\frac{T}{2t_1} + \frac{3}{2}\Big) \tag{2.5}$$

Then

$$\begin{aligned}
(k^2 - j^2 + k + j)t_k &\le (k^2 + 2k - 1)t_k \\
&\le ((k+1)^2 - 2)t_k \\
&\le ((n+1)^2 - 2)T \\
&\le p^* \quad \text{by (2.5)} \\
&\le r.
\end{aligned}$$

□

Proposition 2.5. *For the n future times t_k, $k = 1, 2, 3, \dots$ as in Theorem 2.2, we consider a real number p^* in Lemma 2.4 and $r \ge p^*$. Then*

$$|c_k - \bar{c}_k| \le C(t_1)ke^{-2kt_k}, \quad k = 1, 2, 3, \dots, n.$$

Proof. From the definition of c_k and $\bar{c}_k$, we obtain

$$\begin{aligned}
|c_k - \bar{c}_k| &= \Big|\sum_{j=1}^{k-1}((j+2)^{-r} - c_j)e^{(k+j-1)(k-j)t_k} - \sum_{j=k+1}^{\infty} c_j e^{(k+j-1)(k-j)t_k} \\
&\quad + \sum_{j=k+1}^{n} (j+2)^{-r} e^{(k+j-1)(k-j)t_k}\Big| \\
&\le 2\sum_{j=1}^{k-1}(j+2)^{-r}e^{(k+j-1)(k-j)t_k} + 2\sum_{j=k+1}^{\infty}(j+2)^{-r}e^{(k+j-1)(k-j)t_k} \\
&\le 2\sum_{j=1}^{k-1}(j+2)^{-r}e^{(k+j-1)(k-j)t_k} + 2(k+3)^{-r}\sum_{j=k+1}^{\infty} e^{(k+j-1)(k-j)t_k}
\end{aligned}$$

From the second sum on the right side of the above inequality, we have

$$\begin{aligned}
(k+3)^{-r}\sum_{j=k+1}^{\infty} e^{(k+j-1)(k-j)t_k} &\le (k+3)^{-r}\sum_{j=k+1}^{\infty} e^{-(k+j-1)t_k} \\
&= (k+3)^{-r}\sum_{j=0}^{\infty} e^{-(2k+j)t_k} \\
&= (k+3)^{-r}e^{-2kt_k}\sum_{j=0}^{\infty} e^{-jt_k} \\
&\le (k+3)^{-r}e^{-2kt_k}s(t_1)
\end{aligned} \tag{2.6}$$

where

$$s(t_1) := \sum_{j=0}^{\infty} e^{-jt_1}$$

From Lemma 2.4 and (2.6), we obtain

$$
\begin{aligned}
|c_k-\bar c_k| &\le 2\sum_{j=1}^{k-1}(j+2)^{-r}e^{(k+j-1)(k-j)t_k}+2(k+3)^{-r}\sum_{j=k+1}^{\infty}e^{(k+j-1)(k-j)t_k}\\
&\le e^{-2kt_k}\Big[2\sum_{j=1}^{k-1}(j+2)^{-r}e^{((k+j-1)(k-j)+2k)t_k}+2(k+3)^{-r}s(t_1)\Big]\\
&\le e^{-2kt_k}\Big[2\sum_{j=1}^{k-1}e^{-r}e^{(k^2-j^2+k+j)t_k}+2(k+3)^{-r}s(t_1)\Big] \qquad (2.7)\\
&\le e^{-2kt_k}\big[2(k-1)+\frac{2}{(k+3)^r}s(t_1)\big]\\
&\le ke^{-2kt_k}[2+2s(t_1)]\\
&= C(t_1)ke^{-2kt_k},
\end{aligned}
$$

□

Proof of Theorem 2.2. We define an approximation of $f\in\mathcal{B}_r$ as

$$
\bar f_n(x):=\sum_{k=1}^{n}\hat{\bar f}_k\sin\Big(\frac{2k-1}{2}x\Big)
$$

where $\hat{\bar f}_k=\bar c_k/\sin\Big(\frac{2k-1}{2}x_0\Big)$. By defining $\hat{\bar f}_k=0$ for all $k\ge n+1$, we obtain

$$
\begin{aligned}
\|f-\bar f_n\|^2 &= \Big\|\sum_{k=1}^{\infty}(\hat f_k-\hat{\bar f}_k)\sin\Big(\frac{2k-1}{2}x\Big)\Big\|^2_{L^2([0,\pi])}\\
&\le C_1\sum_{k=1}^{\infty}\big|\hat f_k-\hat{\bar f}_k\big|^2 \quad \text{for some constant } C_1\\
&\le C_1\Big[\sum_{k=1}^{n}\big|\hat f_k-\hat{\bar f}_k\big|^2+\sum_{k=n+1}^{\infty}\big|\hat f_k\big|^2\Big]\\
&\le C_1\Big[\sum_{k=1}^{n}\big|\hat f_k-\hat{\bar f}_k\big|^2+n^{-2r}\sum_{k=n+1}^{\infty}(k+2)^{2r}\big|\hat f_k\big|^2\Big]\\
&\le C_1\Big[\sum_{k=1}^{n}\Big|\frac{c_k}{\sin kx_0}-\frac{\bar c_k}{\sin kx_0}\Big|^2+n^{-2r}\Big]\\
&\le C_2(t_1)\sum_{k=1}^{n}k^2e^{-4kt_k}+C_1n^{-2r} \quad \text{by (1.7) and Proposition 2.5}\\
&\le C_2(t_1)\sum_{k=1}^{n}\frac{k^2}{e^{4k(n+k-1)t_1}}+C_1n^{-2r}\\
&\le \frac{C_2(t_1)}{e^{4nt_1}}\sum_{k=1}^{n}k^2+C_1n^{-2r}\\
&\le \frac{C_2(t_1)}{e^{4nt_1}}\frac{n(n+1)(2n+1)}{6}+C_1n^{-2r}
\end{aligned}
$$

$$= C_2(t_1)\frac{n^3}{6e^{4nt_1}}\big(1+\frac{1}{n}\big)\big(2+\frac{1}{n}\big) + C_1 n^{-2r}$$

For each pair of the parameters r and t_1, the sequence

$$\Big\{\frac{m^{2r+3}}{6e^{4mt_1}}\big(1+\frac{1}{m}\big)\big(2+\frac{1}{m}\big)\Big\}$$

is convergent. So, there exists a constant $c(r, t_1)$ such that

$$\frac{n^{2r+3}}{6e^{4nt_1}}\big(1+\frac{1}{n}\big)\big(2+\frac{1}{n}\big) \le c(r, t_1). \tag{2.8}$$

Thus from (2.8), we have $\|f - \bar{f}_n\|_{L^2}^2 \le C(r, t_1)n^{-2r}$. Therefore,

$$\|f - \bar{f}_n\|_{L^2} \le Cn^{-r},$$

where C is a constant depending on t_1 and r. This completes the proof. □

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