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LIMIT FOR THE p-LAPLACIAN EQUATION WITH DYNAMICAL BOUNDARY CONDITIONS

EYLEM ÖZTÜRK, JULIO D. ROSSI

In memory of Alan C. Lazer, a great mathematician

Abstract. In this article we study the limit as $p \to \infty$ in the evolution problem driven by the p-Laplacian with dynamical boundary conditions. We prove that the natural energy functional associated with this problem converges to a limit in the sense of Mosco convergence and as a consequence we obtain convergence of the solutions to the evolution problems. For the limit problem we show an interpretation in terms of optimal mass transportation and provide examples of explicit solutions for some particular data.

1. Introduction

Our main purpose in this article is to study a nonlinear diffusion equation obtained as the limit as $p \to \infty$ to the p-Laplacian with dynamical boundary conditions. More precisely, we look for the limit as $p \to \infty$ of the solutions to the problem

$$\begin{gathered} 0 = \Delta_p u(x,t), \quad x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial t}(x,t) + |\nabla u|^{p-2}\frac{\partial u}{\partial \eta}(x,t) = f(x,t), \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \partial\Omega. \end{gathered} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\frac{\partial u}{\partial \eta}$ denotes the outer normal derivative of u and f is a nonnegative function that represents a given source term localized on $\partial\Omega$, which is interpreted physically as adding material to an evolving system, within which mass particles are continually rearranged by diffusion. In this model it is assumed that diffusion is much faster inside the domain than on the boundary, hence the time derivative appears only in the boundary condition.

Associated with this evolution problem we have the functional $E_p : L^2(\partial\Omega) \mapsto \mathbb{R} \cup \{+\infty\}$,

$$E_p(u) = \begin{cases} \min\limits_{v \in W^{1,p}(\Omega), \mathrm{trace}(v) = u} \frac{1}{p}\int_\Omega |\nabla v|^p & u \in \mathrm{trace}(W^{1,p}(\Omega)), \\ +\infty & u \notin \mathrm{trace}(W^{1,p}(\Omega)). \end{cases} \tag{1.2}$$

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As we mentioned before our aim is to look for the limit as $p \to \infty$ of the solutions u_p to (1.1). To this end we use a general result by Mosco [23, 24]: if the associated functionals converge to a limit functional (in an adequate sense, that roughly speaking, means convergence of the epigraphs, see Section 2 for the precise definition) then the corresponding solutions to the associated evolution problems converge to the solution associated with the limit functional.

The limit of the functional E_p as $p \to \infty$ is $E_\infty : L^2(\partial\Omega) \mapsto \mathbb{R} \cup \{+\infty\}$, given by

$$E_\infty(u) = \begin{cases} 0 & u \in A_\infty, \\ +\infty & u \notin A_\infty, \end{cases} \tag{1.3}$$

where

$$A_\infty = \big\{ u \in C(\partial\Omega) : \exists v : \overline{\Omega} \mapsto \mathbb{R} \text{ with } |\nabla v| \le 1 \text{a.e } \Omega, v|_{\partial\Omega} = u \big\}.$$

Our first result reads as follows.

Theorem 1.1. *The functionals E_p converge to E_∞ as $p \to \infty$ in the Mosco sense.*

As a consequence we have the convergence of the solutions to our evolution problem (1.1).

Theorem 1.2. *Let $u_p(x,t)$ be the solution of problem (1.1) with a fixed initial condition $u_0 \in \overline{A_\infty}^{L^2(\partial\Omega)}$ and fixed right hand side $f \in L^1(0,T : L^2(\partial\Omega))$. Then*

$$u_p \to u_\infty \tag{1.4}$$

as $p \to \infty$ in $C([0,T] : L^2(\partial\Omega))$, that is,

$$\lim_{p\to\infty} \max_{t\in[0,T]} \|u_p(\cdot,t) - u_\infty(\cdot,t)\|_{L^2(\partial\Omega)} = 0.$$

Moreover, the limit u_∞ is characterized as the solution to

$$\begin{gathered} f(x,t) - \frac{\partial u}{\partial t}(x,t) \in \partial E_\infty(u(x,t)) \quad x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \quad x \in \partial\Omega. \end{gathered} \tag{1.5}$$

If we assume that $u_0 \in L^1(\partial\Omega)$ and f is such that

$$\sup_{t\in[0,T]} \int_{\partial\Omega} |f(x,t)| d\sigma(x) + \int_{\partial\Omega} \Big| \frac{\partial f}{\partial t}(x,t) \Big| d\sigma(x) < +\infty.$$

Then there exists a subsequence $p_i \to \infty$ such that

$$\begin{gathered} u_{p_i} \to u_\infty \quad \text{a.e. and strongly in } L^2(\partial\Omega \times [0,T]), \\ \nabla u_{p_i} \rightharpoonup \nabla u_\infty \quad \text{weakly in } L^2(\partial\Omega \times [0,T]), \\ \frac{\partial u_{p_i}}{\partial t} \rightharpoonup \frac{\partial u_\infty}{\partial t} \quad \text{weakly in } L^2(\partial\Omega \times [0,T]). \end{gathered} \tag{1.6}$$

Finally, we relate the limit problem with an optimal mass transport problem in Theorem 1.3. the optimal mass transport problem is defined with a cost given by the distance between points on $\partial\Omega$ considering paths inside Ω, that is defined as the minimum of the lengths of the paths inside Ω that join the two points. We call this distance d_Ω. It turns out that the limit of the solution to the limit problem, $u_\infty(\cdot,t)$, is a Kantorovich potential for the optimal mass transport problem between $f(\cdot,t)$ and $\frac{\partial u_\infty}{\partial t}(\cdot,t)$.

Theorem 1.3. *The solution to the limit problem* (1.5) *satisfies*

$$\begin{aligned}&\int_{\partial\Omega} u_\infty(x,t)\Big(\frac{\partial u_\infty}{\partial t}(x,t) - f(x,t)\Big)d\sigma(x)\\&= \max_{v:|v(x)-v(y)|\le d_\Omega(x,y)} \int_{\partial\Omega} v(x)\Big(\frac{\partial u_\infty}{\partial t}(x,t) - f(x,t)\Big)d\sigma(x),\end{aligned}$$

that is, u_∞ is a Kantorovich potential for the dual formulation of the Monge-Kantorovich mass transport problem between $f(\cdot,t)d\sigma$ and $\frac{\partial u_\infty}{\partial t}(\cdot,t)d\sigma$.

As was pointed out in [1], the limit problem (1.5) can be interpreted as a model for the formation and growth of a sandpile where particles of sand are distributed on $\partial\Omega$ (here $u_\infty(x,t)$ describes the amount of the sand at the point x at time t). The main assumption being that the sandpile is stable when the slope is less than or equal to one and unstable if not.

We also include some explicit examples of solutions to the limit problem. In these examples one can appreciate the mass transport interpretation of the limit problem. Also we illustrate a curious phenomenon, the support of the solution on $\partial\Omega$ may be disconnected even if the domain is strictly convex, the initial condition is zero and the reaction has connected support.

Dynamical boundary conditions appear in modeling physical phenomena when there is a thin layer around the boundary in which reaction takes place. We refer to [11, 12, 13, 16, 20, 21, 25] for general references concerning evolution problems with this kind of boundary conditions.

As a precedent concerning limits as $p \to \infty$, we mention that problem (1.1) in the elliptic (time independent) case was studied in [17] (see also [18] for the associated eigenvalue problem). Here one needs to assume that $\int_{\partial\Omega} f = 0$ (otherwise, there is no solution) and to have uniqueness of solutions one normalizes according to $\int_{\partial\Omega} u = 0$.

Concerning evolution problems with the p-Laplacian, the counterpart of our results for the Cauchy problem was obtained in [1, 14]. In these references it was studied the limiting behavior as $p \to \infty$ of solutions to the quasilinear parabolic problem

$$\begin{gathered}\frac{\partial v}{\partial t}(x,t) - \Delta_p v(x,t) = f(x,t), \quad \text{in } (0,T)\times\mathbb{R}^N,\\ v(x,0) = u_0(x), \quad \text{in } \mathbb{R}^N.\end{gathered}$$

In [1], assuming that u_0 is a Lipschitz function with compact support, satisfying $|\nabla u_0| \le 1$, it is proved that $v_p \to v_\infty$ and the limit function v_∞ satisfies

$$f(x,t) - \frac{\partial v_\infty}{\partial t}(x,t) \in \partial F_\infty(v_\infty(x,t)),$$

where

$$F_\infty(v) = \begin{cases} 0, & \text{if } |\nabla v| \le 1,\\ +\infty, & \text{otherwise.}\end{cases}$$

Other related papers that deal with limits as $p \to \infty$ in p-Laplacian problems are [4, 5, 6, 19]. The relation between a limit as $p \to \infty$ in a p-Laplacian problem and optimal mass transport was first found in [15] (see also [5]).

The rest of the paper is organized as follows: in Section 2 we gather some preliminary results concerning Mosco convergence of functionals; in Section 3 we prove the convergence of the functionals E_p to E_∞ stated Theorem 1.1 and we deduce the convergence of the solutions to the evolution problems in Theorem 1.2. In Section 4 we deal with the Mass transport interpretation of the limit problem. Finally, in Section 5 we include some explicit examples of solutions to the limit problem.

2. Preliminaries

Now, we recall the definition of Mosco-convergence. If X is a metric space, and $\{A_n\}$ is a sequence of subsets of X, we define

$$\liminf_{n\to\infty} A_n := \big\{x \in X : \exists x_n \in A_n,\ x_n \to x\big\},$$
$$\limsup_{n\to\infty} A_n := \big\{x \in X : \exists x_{n_k} \in A_{n_k},\ x_{n_k} \to x\big\}.$$

If X is a normed space, we denote by s-lim and w-lim the above limits associated, respectively, to the strong and to the weak topology of X.

Definition 2.1. Let H be a Hilbert space. Given $\Psi_n, \Psi : H \to (-\infty, +\infty]$ convex, lower-semicontinuous functionals, we say that Ψ_n converges to Ψ in the sense of Mosco if

$$\text{w-}\limsup_{n\to\infty} \operatorname{Epi}(\Psi_n) \subset \operatorname{Epi}(\Psi) \subset \text{s-}\liminf \operatorname{Epi}(\Psi_n), \tag{2.1}$$

where $\operatorname{Epi}(\Psi_n)$ and $\operatorname{Epi}(\Psi)$ denote the epigraphs of the functionals Ψ_n and Ψ, defined by

$$\operatorname{Epi}(\Psi_n) := \Big\{(u, \lambda) \in L^2(\mathbb{R}^N) \times \mathbb{R} : \lambda \ge \Psi_n(u)\Big\},$$
$$\operatorname{Epi}(\Psi) := \Big\{(u, \lambda) \in L^2(\mathbb{R}^N) \times \mathbb{R} : \lambda \ge \Psi(u)\Big\}.$$

Remark 2.2. We note that (2.1) is equivalent to the requirement that the following two conditions are simultaneously satisfied:

$$\forall u \in D(\Psi)\, \exists u_n \in D(\Psi_n) : u_n \to u \text{ and } \Psi(u) \ge \limsup_{n\to\infty} \Psi_n(u_n), \tag{2.2}$$

$$\text{for every subsequence } \{n_k\},\ \Psi(u) \le \liminf_k \Psi_{n_k}(u_k) \text{ whenever } u_k \rightharpoonup u. \tag{2.3}$$

Here $D(\Psi) := \{u \in H : \Psi(u) < \infty\}$ and $D(\Psi_n) := \{u \in H : \Psi_n(u) < \infty\}$ denote the domains of Ψ and Ψ_n, respectively.

To identify the limit of the solutions u_n to problem (1.1) (see the Introduction), we use methods from Convex Analysis, and so we must first recall some terminology [10, 8, 2].

If H is a real Hilbert space with inner product $(\cdot,\cdot)$ and $\Psi : H \to (-\infty, +\infty]$ is convex, then the subdifferential of Ψ is defined as the multivalued operator $\partial\Psi$ given by

$$v \in \partial\Psi(u) \iff \Psi(w) - \Psi(u) \ge (v, w - u) \quad \forall w \in H.$$

Recall that the epigraph of Ψ is defined by

$$\operatorname{Epi}(\Psi) = \big\{(u, \lambda) \in H \times \mathbb{R} : \lambda \ge \Psi(u)\big\}.$$

Given K, a closed convex subset of H, we define the indicator function of K by

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K. \end{cases}$$

Then the subdifferential is characterized by

$$v \in \partial I_K(u) \iff u \in K \text{ and } (v, w-u) \le 0 \; \forall w \in K.$$

When the convex functional $\Psi : H \to (-\infty, +\infty]$ is proper, lower-semicontinuous, and such that $\min \Psi = 0$, it is well known (see [8]) that the abstract Cauchy problem

$$\begin{gathered} u_t + \partial\Psi(u) \ni f, \quad \text{a.e. } t \in (0,T), \\ u(0) = u_0, \end{gathered}$$

has a unique solution for any $f \in L^1(0,T;H)$ and $u_0 \in \overline{D(\partial\Psi)}$.

The Mosco convergence is a very useful tool to study convergence of solutions of parabolic problems. The following theorem is a consequence of the results in [9, 2].

Theorem 2.3. *Let $\Psi_n, \Psi : H \to (-\infty, +\infty]$ be convex and lower semicontinuous functionals. Then the following two statements are equivalent:*

(i) *Ψ_n converges to Ψ in the sense of Mosco.*
(ii) *$(I + \lambda\partial\Psi_n)^{-1}u \to (I + \lambda\partial\Psi)^{-1}u$ for all $\lambda > 0$, $u \in H$.*

Moreover, either one of the above conditions, (i) or (ii), imply

(iii) *for every $u_0 \in \overline{D(\partial\Psi)}$ and $u_{0,n} \in \overline{D(\partial\Psi_n)}$ such that $u_{0,n} \to u_0$, and for every $f_n, f \in L^1(0,T;H)$ with $f_n \to f$, if $u_n(t)$, $u(t)$ are solutions of the abstract Cauchy problems*

$$\begin{gathered} (u_n)_t + \partial\Psi_n(u_n) \ni f_n \quad \text{a.e. } t \in (0,T) \\ u_n(0) = u_{0,n}, \end{gathered}$$

and

$$\begin{gathered} u_t + \partial\Psi(u) \ni f \quad \text{a.e. } t \in (0,T) \\ u(0) = u_0, \end{gathered}$$

respectively, then $u_n \to u$ in $C([0,T] : H)$.

3. Mosco convergence of the functionals and convergence of the solutions

First, we show some uniform bounds (independent of p) for the solutions u_p to (1.1).

Lemma 3.1. *Fix $T > 0$. Assume that $u_0 \in L^1(\partial\Omega)$ and f is such that*

$$C(f) := \sup_{t\in[0,T]} \int_{\partial\Omega} |f(x,t)| d\sigma(x) + \int_{\partial\Omega} \Big|\frac{\partial f}{\partial t}(x,t)\Big| d\sigma(x) < +\infty. \tag{3.1}$$

Then, there exists a constant C such that

$$\begin{gathered} \sup_{\partial\Omega\times[0,T]} |u_p| \le C, \quad \int_0^T \int_{\partial\Omega} \Big|\frac{\partial u_p}{\partial t}\Big|^2 \le C, \\ \Big(\int_0^T \int_{\partial\Omega} |\nabla u_p|^p\Big)^{1/p} \le C^{1/p}, \end{gathered} \tag{3.2}$$

for every $N + 1 \le p < \infty$. The constant C depends on u_0, $C(f)$ and T.

Proof. In this proof we denote by C a generic constant that depends only on u_0, $C(f)$ and T and may change from one line to another. Now, we use the weak form of (1.1). It holds that

$$\int_0^t \int_{\partial\Omega} \frac{\partial u_p}{\partial t} v + \int_0^t \int_\Omega |\nabla u_p|^{p-2} \nabla u_p \nabla v = \int_0^t \int_{\partial\Omega} f v.$$

Choose a smooth, nondecreasing function $\beta : \mathbb{R} \mapsto \mathbb{R}$ such that $\beta(x) = sgn(x)$ for $|x| \geq \delta > 0$. By approximation we set $v = \beta(u_p)$ as the test function in the weak form of (1.1), to obtain

$$\int_0^t \int_{\partial\Omega} \frac{\partial u_p}{\partial t} \beta(u_p) \leq \int_0^t \int_{\partial\Omega} f \beta(u_p).$$

Hence,

$$\int_{\partial\Omega} B(u_p)(t) - \int_{\partial\Omega} B(u_0) = \int_0^t \int_{\partial\Omega} \frac{\partial B(u_p)}{\partial t} \leq \int_0^t \int_{\partial\Omega} f \beta(u_p),$$

here B satisfies $B'(s) = \beta(s)$. Letting $\delta \to 0$ we obtain

$$\sup_{t\in[0,T]} \int_{\partial\Omega} |u_p|(t) \leq \int_{\partial\Omega} |u_0| + \int_0^T \int_{\partial\Omega} |f| \leq \|u_0\|_{L^1(\partial\Omega)} + C(f)T,$$

where $C(f)$ is the constant that depends on f given in (3.1).

Now, if we take $v = u_p$ as a test function we obtain

$$\int_0^t \int_{\partial\Omega} \frac{\partial u_p}{\partial t} u_p + \int_0^t \int_\Omega |\nabla u_p|^p = \int_0^t \int_{\partial\Omega} f u_p.$$

Since

$$\int_0^t \int_{\partial\Omega} f u_p \leq C(f) \int_0^T \|u_p\|_{L^\infty(\partial\Omega)},$$

we obtain

$$\frac{1}{2} \int_{\partial\Omega} |u_p|^2(t) + \int_0^t \int_\Omega |\nabla u_p|^p \leq \frac{1}{2} \int_{\partial\Omega} |u_0|^2 + C(f) \int_0^T \|u_p\|_{L^\infty(\partial\Omega)}. \tag{3.3}$$

Hence,

$$\begin{aligned} &\sup_{t\in[0,T]} \frac{1}{2} \int_{\partial\Omega} |u_p|^2(t) + \int_0^T \int_\Omega |\nabla u_p|^p \\ &\leq \frac{1}{2} \int_{\partial\Omega} |u_0|^2 + C(f) \int_0^T \|u_p\|_{L^\infty(\partial\Omega)}. \end{aligned} \tag{3.4}$$

Since u_p belongs to $W^{1,p}(\Omega)$, for $p \geq N+1$ we have

$$\begin{aligned} \|u_p(t)\|_{L^\infty(\partial\Omega)} &\leq C\big\{\|\nabla u_p(t)\|_{L^{N+1}(\Omega)} + \|u_p(t)\|_{L^1(\partial\Omega)}\big\} \\ &\leq C\big\{\|\nabla u_p(t)\|_{L^p(\Omega)} + \|u_p(t)\|_{L^1(\partial\Omega)}\big\}. \end{aligned}$$

The constant C in this inequality in independent of p, ($p \geq N+1$), then we have

$$\begin{aligned} \|u_p(t)\|^p_{L^\infty(\partial\Omega)} &\leq C^p\big\{\|\nabla u_p(t)\|^p_{L^p(\Omega)} + \|u_p(t)\|^p_{L^1(\partial\Omega)}\big\} \\ &\leq C^p\big\{\|\nabla u_p(t)\|^p_{L^p(\Omega)} + C^p\big\}. \end{aligned} \tag{3.5}$$

Therefore,

$$\int_0^T \|u_p(s)\|^p_{L^\infty(\partial\Omega)} \leq C^p\Big\{\int_0^T \|\nabla u_p(s)\|^p_{L^p(\Omega)} + C^p T\Big\}.$$

Using (3.4) we obtain

$$\begin{aligned}
&\int_0^T \|u_p(s)\|^p_{L^\infty(\partial\Omega)} \\
&\le C^p\Big\{\frac{1}{2}\int_{\partial\Omega}|u_0|^2 + C(f)\int_0^T \|u_p(s)\|_{L^\infty(\partial\Omega)} + C^pT\Big\} \\
&\le C^p\|u_0\|^2_{L^2(\partial\Omega)} + C^pC(f)\Big(\int_0^T \|u_p(s)\|^p_{L^\infty(\partial\Omega)}\Big)^{1/p}T^{1-1/p} + C^pT \\
&\le C^{p^2/(p-1)} + \frac{1}{2}\int_0^T \|u_p(s)\|^p_{L^\infty(\partial\Omega)}.
\end{aligned}$$

Hence,

$$\Big(\int_0^T \|u_p(s)\|^p_{L^\infty(\partial\Omega)}\Big)^{1/p} \le C.$$

Here the constant C is independent of p. Then, (3.4) implies

$$\Big(\int_0^T\int_\Omega |\nabla u_p|^p\Big)^{1/p} \le C^{1/p}.$$

By an approximation procedure we can use $v = \frac{\partial u}{\partial t}$ as test function to obtain

$$\int_0^T\int_{\partial\Omega}\Big|\frac{\partial u_p}{\partial t}\Big|^2 + \int_0^T\int_\Omega \frac{\partial}{\partial t}\frac{1}{p}|\nabla u_p|^p = \int_0^T\int_{\partial\Omega} f\frac{\partial u_p}{\partial t}.$$

Integrating by parts in time in the last integral, we obtain

$$\begin{aligned}
&\int_0^T\int_{\partial\Omega}\Big|\frac{\partial u_p}{\partial t}\Big|^2 + \int_\Omega \frac{1}{p}|\nabla u_p|^p(T) \\
&= \int_\Omega \frac{1}{p}|\nabla u_0|^p - \int_0^T\int_{\partial\Omega}\frac{\partial f}{\partial t}u_p + \int_{\partial\Omega} f(T)u_p(T) - \int_{\partial\Omega} f(0)u_0.
\end{aligned} \tag{3.6}$$

Hence,

$$\begin{aligned}
\int_\Omega |\nabla u_p|^p(T) &\le \int_\Omega |\nabla u_0|^p + pC(f)\int_0^T \|u_p(s)\|_{L^\infty(\partial\Omega)} \\
&\quad + pC(f)\|u_p(T)\|_{L^\infty(\partial\Omega)} + pC(f)\|u_0\|_{L^\infty(\partial\Omega)} \\
&\le \int_\Omega |\nabla u_0|^p + pC + pC\|u_p(T)\|_{L^\infty(\partial\Omega)}.
\end{aligned}$$

Now, using (3.5) we obtain

$$\begin{aligned}
\|u_p(T)\|^p_{L^\infty(\partial\Omega)} &\le C^p\Big(\int_\Omega |\nabla u_0|^p + pC + pC\|u_p(T)\|_{L^\infty(\partial\Omega)}\Big) + C^p \\
&\le \frac{1}{2}\|u_p(T)\|^p_{L^\infty(\partial\Omega)} + (C^ppC)^{p/(p-1)} + C^p
\end{aligned}$$

and then we conclude that $\|u_p(T)\|_{L^\infty(\partial\Omega)} \le C$. As T is any time we obtain

$$\sup_{t\in[0,T]} \|u_p(t)\|_{L^\infty(\partial\Omega)} \le C.$$

Finally, since $|\nabla u_0| \le 1$, from (3.6) we conclude that

$$\int_0^T\int_{\partial\Omega}\Big|\frac{\partial u_p}{\partial t}\Big|^2 \le C$$

This completes the proof. □

Now, we prove that the functionals E_p converge in the sense of Mosco to the limit functional E_∞.

Proof of Theorem 1.1. First, we want to show that (2.2) holds, that is,

$$\forall u \in D(E_\infty)\ \exists u_p \in D(E_p) : u_p \to u \text{ and } E(u) \geq \limsup_{n\to\infty} E_p(u_p). \tag{3.7}$$

Given $u \in D(E_\infty)$, that is, $u \in A_\infty$, we just take $u_p \equiv u$ as the desired sequence. We clearly have $u_p \to u$ strongly in $L^2(\partial\Omega)$.

Now, from the fact that $u \in A_\infty$, there exists $v^* : \overline{\Omega} \mapsto \mathbb{R}$ with $|\nabla v^*| \leq 1$ a.e Ω and $v^*|_{\partial\Omega} = u$. Hence, we obtain that $u \in D(E_p)$, that is, $u \in trace(W^{1,p}(\Omega)$ and

$$E_p(u_p) = \min_{v\in W^{1,p}(\Omega), trace(v)=u} \frac{1}{p}\int_\Omega |\nabla v|^p \leq \frac{1}{p}\int_\Omega |\nabla v^*|^p \leq \frac{1}{p}|\Omega| \to 0$$

as $p \to \infty$. Then we have

$$0 = E(u) \geq \limsup_{n\to\infty} E_p(u_p) = 0$$

as we wanted to show.

Now, we have to prove that (2.3) also holds, namely, for every subsequence $\{p_k\}$,

$$E_\infty(u) \leq \liminf_k E_{p_k}(u_k) \quad \text{whenever } u_k \rightharpoonup u. \tag{3.8}$$

To see this, we first observe that when $u \in A_\infty = D(E_\infty)$ we have $E_\infty(u) = 0$ and we trivially obtain $E_\infty(u) \leq \liminf_k E_{p_k}(u_k)$ since $E_{p_k}(u_k) \geq 0$.

Also, we can assume that $\liminf_k E_{p_k}(u_k) < +\infty$ (otherwise the desired inequality holds trivially). Hence, for a subsequence we have that there is a constant C such that

$$\frac{1}{p_k} \min_{v\in W^{1,p_k}(\Omega), trace(v)=u_k} \int_\Omega |\nabla v|^{p_k} \leq C.$$

Call v_k a function in $W^{1,p_k}(\Omega)$ that attains the minimum. For this v_k we have

$$\Big(\int_\Omega |\nabla v_k|^{p_k}\Big)^{1/p_k} \leq (p_k C)^{1/p_k}.$$

Now, for $2 < q < \infty$, we obtain

$$\Big(\int_\Omega |\nabla v_k|^q\Big)^{1/q} \leq |\Omega|^{(p_k-q)/p_k q}\Big(\int_\Omega |\nabla v_k|^{p_k}\Big)^{1/p_k} \leq |\Omega|^{(p_k-q)/p_k q}(p_k C)^{1/p_k}.$$

The right-hand side is bounded and hence we can take the limit as $p_k \to \infty$ to obtain that $v_k \rightharpoonup v^*$ weakly in $W^{1,q}(\Omega)$. This limit v^* satisfies

$$\Big(\int_\Omega |\nabla v^*|^q\Big)^{1/q} \leq |\Omega|^{1/q}.$$

Hence, taking $q \to \infty$ we conclude that $v^* \in W^{1,\infty}(\Omega)$ and $|\nabla v^*| \leq 1$ a.e. in Ω.

Now, from the weak convergence of v_k to v^* in $W^{1,q}(\Omega)$ using the Sobolev trace embedding we obtain that $u_k = trace(v_k) \to u = trace(v^*)$ strongly in $L^2(\partial\Omega)$ and hence we have that $u \in A_\infty = D(E_\infty)$. Then, we have

$$0 = E_\infty(u) \leq \liminf_k E_{p_k}(u_k)$$

since $E_{p_k}(u_k) \geq 0$, as we wanted to show. □

As a consequence we obtain the convergence of the corresponding solutions to the associated evolution problems.

Proof of Theorem 1.2. We can apply Theorem 2.3 to obtain the first part of the result, namely, $u_p \to u_\infty$ as $p \to \infty$ in $C([0,T] : L^2(\partial\Omega))$ and the limit u_∞ is characterized as the solution to the limit problem (1.5).

To complete the proof we observe that, from the uniform bounds obtained in Lemma 3.1, we obtain the existence of a subsequence $p_i \to \infty$ such that the convergences stated in (1.6) hold. □

4. Mass transport interpretation of the limit problem

We relate the limit problem with an optimal mass transport problem with a cost given by the distance between points inside Ω that is defined as the infimum of the lengths of curves going from x to y, that is,

$$d_\Omega(x,y) = \inf_{\gamma(0)=x,\gamma(1)=y} \text{length}(\gamma(t)).$$

When the domain Ω is convex the distance d_Ω coincides with the Euclidean distance, we have $d_\Omega(x,y) = |x-y|$.

Given two measures μ, ν on $\partial\Omega$ with the same total mass we consider the transport cost (Monge-Kantorovich mass transport problem)

$$C(\mu,\nu) = \min_{\theta(x,y):\theta|_x=\mu,\theta|_y=\nu} \int_{\partial\Omega\times\partial\Omega} d_\Omega(x,y)d\theta(x,y).$$

Here by $\theta|_x$ we denote the first marginal of θ, that is, $\theta|_x(E) = \theta(E\times\partial\Omega)$ (and similarly with $\theta|_y$ we denote the second marginal of θ).

Associated with an optimal mass transport problem we have its dual formulation that is given by

$$C(\mu,\nu) = \max_{v:|v(x)-v(y)|\le d_\Omega(x,y)} \int_{\partial\Omega} v(x)(d\mu(x) - d\nu(x)).$$

Maximizers of the dual problem are called Kantorovich potentials for the optimal mass transport problem.

It turns out that the limit of the solutions, $u_\infty(\cdot,t)$, is a Kantorovich potential for the optimal mass transport problem between $f(\cdot,t)$ and $\frac{\partial u_\infty}{\partial t}(\cdot,t)$.

Proof of Theorem 1.3. First, let us prove that the limit function u_∞ is admissible for the dual problem. Given two points $x,y\in\partial\Omega$, using that $|\nabla u_\infty(\cdot,t)|\le 1$ a.e. in Ω, we have

$$\begin{aligned}
|u_\infty(x,t) - u_\infty(y,t)| &= \Big|\int_0^1 \frac{\partial u_\infty(\gamma(s),t)}{\partial s}(s)ds\Big| \\
&= \Big|\int_0^1 \langle\nabla u_\infty(\gamma(s),t),\gamma'(s)\rangle ds\Big| \\
&\le \text{length}(\gamma(s))
\end{aligned}$$

and hence we obtain

$$|u_\infty(x) - u_\infty(y)| \le d_\Omega(x,y).$$

Now, we show that in fact $u_\infty(\cdot,t)$ is a solution to the dual problem. We have that $u_\infty(x,t)$ solves the limit equation

$$f(x,t) - \frac{\partial u_\infty}{\partial t}(x,t) \in \partial E_\infty(u(x,t)),$$

that is,

$$E_\infty(v(x)) \ge E_\infty(u_\infty(x,t)) + \int_{\partial\Omega} \Big(f(x,t) - \frac{\partial u_\infty}{\partial t}(x,t)\Big)(v(x) - u_\infty(x,t))$$

Take $v \in A_\infty$. Since $u_\infty(\cdot,t) \in A_\infty$ we have

$$0 \ge \int_{\partial\Omega} \Big(f(x,t) - \frac{\partial u_\infty}{\partial t}(x,t)\Big)(v(x) - u_\infty(x,t))$$

and therefore,

$$\int_{\partial\Omega} u_\infty(x,t)\Big(\frac{\partial u_\infty}{\partial t}(x,t) - f(x,t)\Big) \ge \int_{\partial\Omega} v(x)\Big(\frac{\partial u_\infty}{\partial t}(x,t) - f(x,t)\Big),$$

for every v such that $|v(x) - v(y)| \le d_\Omega(x,y)$.

We have obtained that $u_\infty(\cdot,t)$ is a Kantorovich potential for the optimal mass transport problem between $f(\cdot,t)d\sigma$ and $\frac{\partial u_\infty}{\partial t}(\cdot,t)d\sigma$. □

5. Examples

In this final section we include some simple examples in which one can find the solution to the limit evolution problem

$$\begin{gathered} f(x,t) - \frac{\partial u_\infty}{\partial t}(x,t) \in \partial E_\infty(u_\infty(x,t)) \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \quad x \in \partial\Omega. \end{gathered} \tag{5.1}$$

Example 5.1. We consider the 1-dimensional case with $\Omega = (0,1)$, and take

$$f(x,t) = \begin{cases} 0, & x = 0,\ t > 0, \\ 1, & x = 1,\ t > 0, \end{cases}$$

(notice that f is defined on $\partial\Omega \times (0,T)$) and $u_0 \equiv 0$. Then we have

$$u_\infty(x,t) = \begin{cases} 0, & x = 0,\ 0 \le t \le 1, \\ t, & x = 1,\ 0 \le t \le 1, \end{cases}$$

and

$$u_\infty(x,t) = \begin{cases} \frac{1}{2}(t-1), & x = 0,\ 1 \le t, \\ \frac{1}{2}(t-1) + 1, & x = 1,\ 1 \le t. \end{cases}$$

Notice that

$$|u_\infty(1,t) - u_\infty(0,t)| \le 1 = d_\Omega(0,1) = 1, \quad \text{for every } t \ge 0.$$

Also we remark that the solution starts to grow at $x = 1$ with $\frac{\partial u_\infty}{\partial t}(1,t) = 1$ until it reaches $u_\infty(1,t_0) = 1$ (this happens at $t_0 = 1$) and next it grows at the slower rate $\frac{\partial u_\infty}{\partial t}(1,t) = 1/2$ (but also grows at $x = 0$ with $\frac{\partial u_\infty}{\partial t}(0,t) = 1/2$). This is due to the fact that the unit mass added at $x = 1$ is divided between two locations $x = 0$ and $x = 1$ in order to keep the constraint $|u_\infty(1,t) - u_\infty(0,t)| \le 1$ for times $t \ge 1$.

Example 5.2. We consider a nontrivial initial condition for the setting of Ω, and $f(x,t)$. Note that f is defined on $\partial\Omega \times (0,T))$ and fix a nonnegative C^1 initial condition u_0 with $|u_0'(x)| \le 1$ for $x \in [0,1]$. Then we have

$$u_\infty(x,t) = \begin{cases} u_0(0), & x = 0,\ 0 \le t \le t_0, \\ u_0(1) + t, & x = 1,\ 0 \le t \le t_0, \end{cases}$$

with t_0 the first time at which $u_0(1) + t_0 - u_0(0) = 1$, that is

$$t_0 = u_0(0) - u_0(1) + 1.$$

Note that $t_0 \ge 0$ because $u_0(0) - u_0(1) + 1 = u_0'(\xi) + 1 \ge 0$. Also note that $u_\infty(x,t) \in A_\infty$, since there exists a function v with $|v'| \le 1$ in $[0,1]$ such that $v(0) = u_0(0)$, $v(1) = u_0(1) + t$ (in addition, this function v can be chosen satisfying $v \ge u_0$ in $[0,1]$).

For times larger than t_0 we have

$$u_\infty(x,t) = \begin{cases} u_0(0) + \frac{1}{2}(t - t_0), & x = 0,\ t_0 \le t, \\ u_0(1) + \frac{1}{2}(t - t_0) + t_0, & x = 1,\ t_0 \le t. \end{cases}$$

Example 5.3. Now, we extend above exmaples to several dimensions. Take a fixed domain $\Omega \subset \mathbb{R}^N$, fix a subdomain of its boundary $\Gamma \subset \partial\Omega$ and consider $f : \partial\Omega \times (0,T) \mapsto \mathbb{R}^N$,

$$f(x,t) = \chi_\Gamma(x)$$

and, as before, $u_0 \equiv 0$. We remark that Example 5.1 is a particular case of this more general setting.

In this case the solution $u_\infty(x,t)$ to the limit problem is

$$u_\infty(x,t) = (a(t) - d_\Omega(x,\Gamma))_+,$$

with $a(t)$ the solution to the ODE

$$a'(t)\Big|\{x \in \partial\Omega : d_\Omega(x,\Gamma) < a(t)\}\Big|_{H^{N-1}} = |\Gamma|_{H^{N-1}},$$
$$a(0) = 0.$$

Here we denoted by $|E|_{H^{N-1}}$ the $N-1$-dimensional surface measure of a measurable set $E \subset \partial\Omega$.

Notice that the support of $u_\infty(\cdot,t)$ in $\partial\Omega$ can be disconnected even if the domain is strictly convex and the set where the source is localized Γ is connected. In fact, this is the case the set

$$\{x \in \partial\Omega : d_\Omega(x,\Gamma) < k\}$$

is disconnected for some $k > 0$. Also notice that, since Ω is bounded and $\partial\Omega$ is smooth (it has finite H^{N-1}–measure), there exists a finite time t_0 such that the support of $u_\infty(\cdot,t)$ is the whole $\partial\Omega$ for times $t \ge t_0$. At this time t_0 we have

$$a(t_0) = \max_{x\in\partial\Omega} d_\Omega(x,\Gamma)$$

and then we have

$$u_\infty(x,t) = \Big(\max_{x\in\partial\Omega} d_\Omega(x,\Gamma) - dist(x,\Gamma)\Big),$$

After this time the solution is

$$u_\infty(x,t) = \Big(\frac{|\Gamma|_{H^{N-1}}}{|\partial\Omega|_{H^{N-1}}} t + \max_{x\in\partial\Omega} d_\Omega(x,\Gamma) - dist(x,\Gamma)\Big),$$

That is, after t_0 the solution grows uniformly in the whole $\partial\Omega$ with speed $\frac{|\Gamma|_{H^{N-1}}}{|\partial\Omega|_{H^{N-1}}}$.

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Eylem Öztürk
Department of Mathematics, Hacettepe University, 06800 Beytepe, Ankara, Turkey
Email address: eyturk1983@gmail.com

Julio D. Rossi
Departamento de Matemática, FCEyN, Universidad de Buenos Aires, Pabellon I, Ciudad Universitaria (C1428BCW), Buenos Aires, Argentina
Email address: jrossi@dm.uba.ar