Special Issue in honor of Alan C. Lazer
Electronic Journal of Differential Equations, Special Issue 01 (2021), pp. 149–167.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu or https://ejde.math.unt.edu

OSCILLATORY SOLUTIONS AND CRITICAL EXPONENTS FOR FULLY NONLINEAR EQUATIONS

FILOMENA PACELLA, DAVID STOLNICKI

Abstract. We study existence and nonexistence of radial positive solutions for a class of fully nonlinear equations involving Pucci's extremal operators. By analyzing the periodic orbits of an associated dynamical system we are able to give estimates on the range of the exponents for which entire oscillating solutions exist. This, in turn, allows to improve previous bounds on the critical exponents defined in [2].

1. Introduction

In this article we study positive radial solutions of the fully nonlinear Lane Emden equation

$$-\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) = u^p \quad \text{in } \Omega \subseteq \mathbb{R}^N,\ N \geq 3 \tag{1.1}$$

where Ω is either the whole $\mathbb{R}^N$ or the ball B_R of radius $R > 0$, centered at the origin. In the last case, we assume homogeneous Dirichlet boundary condition. We recall that for a $\mathcal{C}^2$ function in $\mathbb{R}^N$, $N \geq 2$, the Pucci's operators are defined by

$$\mathcal{M}^{+}_{\lambda,\Lambda}(D^2u) = \Lambda \sum_{\mu_i>0} \mu_i + \lambda \sum_{\mu_i<0} \mu_i$$
$$\mathcal{M}^{-}_{\lambda,\Lambda}(D^2u) = \lambda \sum_{\mu_i>0} \mu_i + \Lambda \sum_{\mu_i<0} \mu_i$$

where $0 < \lambda \leq \Lambda$ are the ellipticity constants and $\mu_i = \mu_i(D^2u)$, $i = 1, \dots, N$ are the eigenvalues of the hessian matrix D^2u. Associated with $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ are dimension like numbers $\tilde{N}_+$ and $\tilde{N}_-$ defined as

$$\tilde{N}_+ = \frac{\lambda}{\Lambda}(N-1) + 1, \quad \tilde{N}_- = \frac{\Lambda}{\lambda}(N-1) + 1 \tag{1.2}$$

These numbers allow us to give estimates for the exponent p for which existence or nonexistence of radial solutions of (1.1) in B_R or $\mathbb{R}^N$ holds when $N \geq 3$ and $\tilde{N}_+ > 2$, (note that $\tilde{N}_-$ is always larger than two if $N \geq 3$).

Indeed a first result obtained in [1] shows that if $N \geq 3$ and $p \leq \frac{\tilde{N}_\pm}{\tilde{N}_\pm - 2}$, then no nontrivial positive viscosity supersolutions of (1.1) exist in $\mathbb{R}^N$. Using this result,

2010 *Mathematics Subject Classification.* 35J15, 35J60, 35B09, 34C25.
Key words and phrases. Fully nonlinear equations; radial solutions; critical exponents; dynamical systems.

Published October 6, 2021.

the existence of positive solutions in bounded domains, not necessarily radial, was proved in [6] for the same range of exponents.

In the radial setting Felmer and Quaas [2] (see also [3]) provided, for $N \geq 3$, the existence of critical exponents p_+^* for $\mathcal{M}^+_{\lambda,\Lambda}$ and p_-^* for $\mathcal{M}^-_{\lambda,\Lambda}$ satisfying, for $\lambda < \Lambda$,

$$\begin{gathered} \max\Big\{\frac{\tilde{N}_+}{\tilde{N}_+ - 2}, \frac{N+2}{N-2}\Big\} < p_+^* < \frac{\tilde{N}_+ + 2}{\tilde{N}_+ - 2}, \\ \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2} < p_-^* < \frac{N+2}{N-2} \end{gathered} \tag{1.3}$$

which are thresholds for the existence of radial positive solutions of (1.1) in the ball or in $\mathbb{R}^N$. To state precisely their result, let us recall some definitions.

Definition 1.1. Let u be a radial solution of (1.1) in $\mathbb{R}^N$ and $\alpha = \frac{2}{p-1}$. Then u is said to be:

(i) fast decaying if there exists $c > 0$ such that $\lim_{r\to\infty} r^{\tilde{N}_- - 2} u(r) = c$ (or $\lim_{r\to\infty} r^{\tilde{N}_+ - 2} u(r) = c$),
(ii) slow decaying if there exists $c > 0$ such that $\lim_{r\to\infty} r^{\alpha} u(r) = c$.
(iii) pseudo slow decaying if there exist constants $0 < c_1 < c_2$ such that
$$c_1 = \liminf_{r\to\infty} r^{\alpha} u(r) < \limsup_{r\to\infty} r^{\alpha} u(r) = c_2\,.$$

The definitions (i) and (ii) are standard in the theory of classical semilinear Lane Emden equations. Instead (iii) was introduced in [2] and corresponds to solutions oscillating at $+\infty$ by changing concavity infinitely many times.

Theorem 1.2 ([2]). *Assume tildeN$_+ > 2$ and $\lambda < \Lambda$, then*

(1) *if $p \in (1, p_\pm^*)$ there is no positive solution of (1.1) in $\mathbb{R}^N$;*
(2) *if $p = p_\pm^*$ there exists a unique positive fast decaying solution of (1.1) in $\mathbb{R}^N$;*
(3) *if $p > p_\pm^*$ there exists a unique positive solution of (1.1) in $\mathbb{R}^N$, which is either slow decaying or pseudo-slow decaying;*
(4) *a positive solution in B_R exists (and is unique) if and only if $p \in (1, p_\pm^*)$.*

In (2) and (3) uniqueness is meant up to scaling.

In addition, in the case of $\mathcal{M}^+_{\lambda,\Lambda}$, the precise range of exponents p for which oscillating pseudo-slow decaying solutions exist, is provided in [2].

The proof of Theorem 1.2 given in [2] is quite long and technically complicated. It is a combination of the Emden-Fowler phase plane analysis and the Coffman Kolodner technique. An alternative simpler proof, which also extends to weighted equations and singular solutions is given in [4]. It is entirely based on the study of the orbits of an associated quadratic dynamical system. The same approach has been used in [5] to define and give bounds for a critical exponent that can be defined for the operator $\mathcal{M}^-_{\lambda,\Lambda}$ in dimension $N = 2$.

While analyzing the dynamical system (defined in Section 2) it is important to understand for which values of p periodic orbits exist. Indeed, this enters not only in detecting the presence of pseudo-slow decaying solutions of (1.1) but also to study the existence of fast decaying solutions. This idea was exploited in [5] to estimate the critical exponent in dimension two.

In this article we refine the results obtained in [2] and [4] by extending the approach of [5] to higher dimensions. In this way, we obtain, on the one hand, information on the exponents p for which oscillating pseudo-slow decaying solutions do not exist; on the other hand, a better estimate of the critical exponents $p^*_\pm$, as compared with [2]. More precisely we prove the following theorems.

Theorem 1.3 ($\mathcal{M}^+_{\lambda,\Lambda}$ case)**.** *Let*

$$\hat{a}_+ = \frac{\left(\frac{N}{N-1}\right)^{\frac{N-2}{2}} - 1}{1 - \left(\frac{\Lambda(\tilde{N}_+-2)}{\lambda(N-1)}\right)^{\frac{\tilde{N}_+-2}{2}}}. \tag{1.4}$$

If

$$p \le \frac{\hat{a}_+(N+2) + (\tilde{N}_+ + 2)}{\hat{a}_+(N-2) + (\tilde{N}_+ - 2)} = h(\hat{a}_+), \tag{1.5}$$

then problem (1.1) *for* $\mathcal{M}^+_{\lambda,\Lambda}$ *does not have any pseudo-slow decaying solution. Moreover*

$$p^*_+ \ge h(\hat{a}_+) > \frac{N+2}{N-2}\,. \tag{1.6}$$

Theorem 1.4 ($\mathcal{M}^-_{\lambda,\Lambda}$ case)**.** *Let*

$$\hat{a}_- = \frac{\left(\frac{N}{N-1}\right)^{\frac{N-2}{2}} - 1}{1 - \left(\frac{\lambda(\tilde{N}_--2)}{\Lambda(N-1)}\right)^{\frac{\tilde{N}_--2}{2}}}\,. \tag{1.7}$$

If

$$p \ge \frac{\hat{a}_-(N+2) + (\tilde{N}_- + 2)}{\hat{a}_-(N-2) + (\tilde{N}_- - 2)} = h(\hat{a}_-)\,, \tag{1.8}$$

then problem (1.1) *for* $\mathcal{M}^-_{\lambda,\Lambda}$ *does not have any pseudo-slow decaying solution. Moreover*

$$p^*_- \le h(\hat{a}_-) < \frac{N+2}{N-2}\,. \tag{1.9}$$

Although not so simple to write, the numbers $h(\hat{a}_+)$ and $h(\hat{a}_-)$ give estimates for the critical exponents substantially better than (1.3). This can be easily seen by assigning values to N and to the ellipticity constant λ, Λ (see Section 5).

Even better estimates could be obtained with different choices of certain sets related to the periodic orbits of the dynamical systems (see Corollary 3.3, Corollary 3.6 and Section 5).

The article is organized as follows. In Section 2 we study the dynamical system associated to (1.1), while in Section 3, we prove results about its periodic orbits. In Section 4 we prove Theorem 1.3 and Theorem 1.4. In Section 5 we show some comparisons between our estimates for the critical exponents and the one obtained in [2] and [4].

2. Associated dynamical systems

Since we consider radial solutions to (1.1) we write $u = u(r) = u(|x|)$ as an expression of a function u in radial coordinates. Then, the eigenvalues of D^2u are the simple eigenvalue $u''(r)$ and, with multiplicity $(N-1)$, $\frac{u'(r)}{r}$ (see [2]).

Thus u satisfies a corresponding ODE from which it is easy to deduce that u is decreasing as long as it is positive, concave in an interval $(0, \tau_0)$ and changes concavity at least once (see [2] or [4]):

$$\begin{aligned} u'' &= M(-\lambda(N-1)r^{-1}u' - u^p), \text{ for } \mathcal{M}^+_{\lambda,\Lambda} \\ u'' &= m(-\Lambda(N-1)r^{-1}u' - u^p) \text{ for } \mathcal{M}^-_{\lambda,\Lambda}, \end{aligned} \tag{2.1}$$

where

$$M(s) = \begin{cases} s/\Lambda & \text{if } s \le 0 \\ s/\lambda & \text{if } s > 0, \end{cases} \tag{2.2}$$

and

$$m(s) = \begin{cases} s/\lambda & \text{if } s \le 0 \\ s/\Lambda & \text{if } s > 0, \end{cases} \tag{2.3}$$

Next we introduce the auxiliary functions

$$X(t) = -\frac{ru'(r)}{u(r)}, \quad Z(t) = -\frac{ru(r)^p}{u'(r)} \quad \text{for } t = \ln r \tag{2.4}$$

whenever $u(r) \neq 0$, $u'(r) \neq 0$.

Since $u > 0$, and $u' < 0$, the above quantities belong to the first quadrant of the (X, Z) plane. In the new variables, the equations (2.1) become the following autonomous dynamical system

$$(\dot{X}, \dot{Z}) = F(X, Z) = \big(f(X, Z), g(X, Z)\big), \tag{2.5}$$

where the dot $\dot{}$ stands for derivation with respect to t, and f, g are given by for $\mathcal{M}^+_{\lambda,\Lambda}$:

$$f(X, Z) = \begin{cases} X(X - (N-2) + \frac{Z}{\lambda}) & \text{if } (X, Z) \in R^+_\lambda \\ X(X - (\tilde{N}_+ - 2) + \frac{Z}{\Lambda}) & \text{if } (X, Z) \in R^-_\lambda, \end{cases} \tag{2.6a}$$

$$g(X, Z) = \begin{cases} Z(N - pX - \frac{Z}{\lambda}) & \text{if } (X, Z) \in R^+_\lambda \\ Z(\tilde{N}_+ - pX - \frac{Z}{\Lambda}) & \text{if } (X, Z) \in R^-_\lambda, \end{cases} \tag{2.6b}$$

(2.6c)

and for $\mathcal{M}^-_{\lambda,\Lambda}$:

$$f(X, Z) = \begin{cases} X(X - (N-2) + \frac{Z}{\Lambda}) & \text{if } (X, Z) \in R^+_\Lambda \\ X(X - (\tilde{N}_- - 2) + \frac{Z}{\lambda}) & \text{if } (X, Z) \in R^-_\Lambda, \end{cases} \tag{2.6d}$$

$$g(X, Z) = \begin{cases} Z(N - pX - \frac{Z}{\Lambda}) & \text{if } (X, Z) \in R^+_\Lambda \\ Z(\tilde{N}_- - pX - \frac{Z}{\lambda}) & \text{if } (X, Z) \in R^-_\Lambda \end{cases} \tag{2.6e}$$

where the regions R^+_λ, R^-_λ, R^+_Λ and R^-_Λ are:

$$R^+_\lambda = \{(X, Z) : Z > \lambda(N-1)\}, \tag{2.7}$$

$$R^-_\lambda = \{(X, Z) : 0 < Z < \lambda(N-1)\}, \tag{2.8}$$

with R^+_Λ and R^-_Λ defined analogously.

Note that whenever $(X(t), Z(t))$ belongs to the lines

$$\ell_+ = \{(X, Z) : Z = \lambda(N-1)\} \quad \text{for } \mathcal{M}^+_{\lambda,\Lambda} \tag{2.9}$$

$$\ell_- = \{(X, Z) : Z = \Lambda(N-1)\} \quad \text{for } \mathcal{M}^-_{\lambda,\Lambda}. \tag{2.10}$$

Then the corresponding solution u of (2.1) satisfies $u''(r) = 0$, $r = e^t$. Hence R^+_λ, R^+_Λ and R^-_λ, R^-_Λ represent, in terms of the new variables (X, Z), the regions of concavity and convexity of u, respectively.

The dynamical system (2.5) was studied in [4] with full details. We recall here the main proprieties which are needed in the sequel, referring the reader to [4, Section 2] for the proofs and further results.

Other important sets which are relevant to study the dynamics induced by (2.5) are:

$$\ell_1^+ = \{(X,Z) : Z = \Lambda(\tilde{N}_+ - 2 - X)\} \quad \text{for } \mathcal{M}^+_{\lambda,\Lambda}, \tag{2.11}$$

$$\ell_1^- = \{(X,Z) : Z = \lambda(\tilde{N}_- - 2 - X)\} \quad \text{for } \mathcal{M}^-_{\lambda,\Lambda}, \tag{2.12}$$

which is the set where $\dot{X} = 0$. Notice that ℓ_1^+ (analogously ℓ_1^-) is a segment entirely contained in R^-_λ (or R^-_Λ in the case of ℓ_1^-), since there are no other points in $1Q$, where $\dot{X} = 0$ in the interior of the region R^+_λ (R^+_Λ for ℓ_1^-).

We also introduce

$$\ell_2^+ = \{(X,Z) \in R^+_\lambda : Z = \lambda(N - pX)\} \cup \{(X,Z) \in R^-_\lambda : Z = \Lambda(\tilde{N}_+ - pX)\}$$

for $\mathcal{M}^+_{\lambda,\Lambda}$, and

$$\ell_2^- = \{(X,Z) \in R^+_\Lambda : Z = \Lambda(N - pX)\} \cup \{(X,Z) \in R^-_\Lambda : Z = \lambda(\tilde{N}_- - pX)\}$$

for $\mathcal{M}^-_{\lambda,\Lambda}$, which is the set where $\dot{Z} = 0$.

Since our aim is to improve the bounds for the critical exponents obtained in [2] we restrict our attention to p greater than $\frac{\tilde{N}_+}{\tilde{N}_+ - 2}$ or $\frac{\tilde{N}_-}{\tilde{N}_- - 2}$.

Proposition 2.1. *The ODE system* (2.5) *admits the following stationary points. For $\mathcal{M}^+_{\lambda,\Lambda}$:*

- (i) $N_0 = (0, N\lambda)$ *which is a saddle point,*
- (ii) $A_0 = (\tilde{N}_+ - 2, 0)$ *which is a saddle point for* $p > \frac{\tilde{N}_+}{\tilde{N}_+ - 2}$,
- (iii) $M_0 = \left(\frac{2}{p-1}, \Lambda(\tilde{N}_+ - 2 - \frac{2}{p-1})\right)$ *which is a source for* $\frac{\tilde{N}_+}{\tilde{N}_+ - 2} < p < \frac{\tilde{N}_+ + 2}{\tilde{N}_+ - 2}$, *a center for* $p = \frac{\tilde{N}_+ + 2}{\tilde{N}_+ - 2}$, *a sink for* $p > \frac{\tilde{N}_+ + 2}{\tilde{N}_+ - 2}$,
- (iv) $O = (0, 0)$ *which is a saddle point.*

For $\mathcal{M}^-_{\lambda,\Lambda}$:

- (i) $N_0 = (0, N\Lambda)$ *which is a saddle point,*
- (ii) $A_0 = (\tilde{N}_- - 2, 0)$ *which is a saddle point for* $p > \frac{\tilde{N}_-}{\tilde{N}_- - 2}$,
- (iii) $M_0 = \left(\frac{2}{p-1}, \lambda(\tilde{N}_- - 2 - \frac{2}{p-1})\right)$ *which is a source for* $\frac{\tilde{N}_-}{\tilde{N}_- - 2} < p < \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2}$, *a center for* $p = \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2}$, *a sink for* $p > \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2}$,
- (iv) $O = (0, 0)$ *which is a saddle point.*

The stationary points and the direction of the vector field F on the relevant sets for (2.5) are displayed in Figure 1.

The set of limit points of a trajectory $\tau(t)$ of the system (2.5), as $t \to -\infty$, is usually called its α-limit and denoted by $\alpha(\tau)$. Analogously it is defined the ω-limit $\omega(\tau)$ at $+\infty$. The precise correspondence between the solutions of (2.1) and the trajectories of (2.5) has been analyzed in [4, Section 3]. We recall here what is relevant to study the entire positive solutions or the ones in a ball of (2.1).

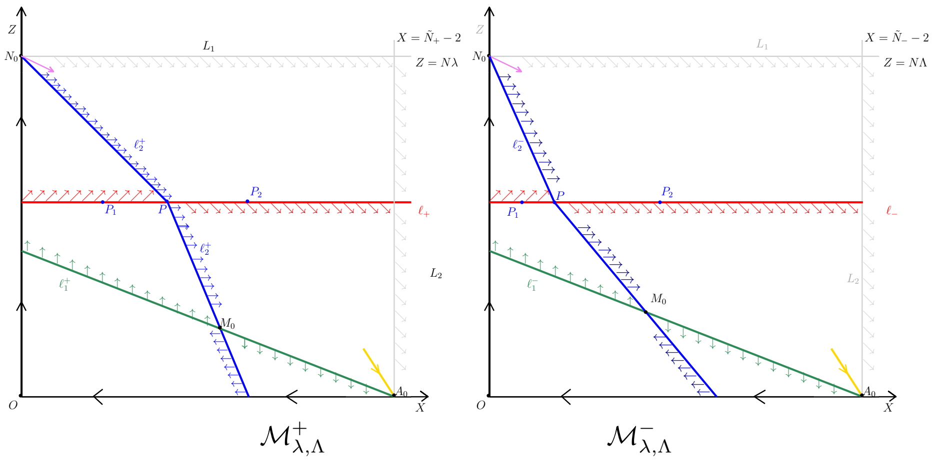

FIGURE 1. Vector field F and the stationary points.

Proposition 2.2. *For every $p > 1$, any regular solution u_p of (2.1) satisfying the initial conditions: $u_p(0) = \gamma > 0$, $u_p'(0) = 0$, corresponds to the unique trajectory Γ_p of (2.5) whose α-limit is the stationary point N_0. Moreover:*

(i) *if $u_p = u_p(r)$ is positive $\forall r \in (0, +\infty)$ then Γ_p is bounded and remains in the rectangle $(0, \tilde{N}_+ - 2) \times (0, N\lambda)$, for all time. ($(0, \tilde{N}_- - 2) \times (0, N\Lambda)$ in case of $\mathcal{M}^-_{\lambda,\Lambda}$)*

(ii) *if there exists R_p such that $u_p(R_p) = 0$ and $u_p > 0$ in $(0, R_p)$, then the corresponding trajectory $\Gamma_p = (X_p, Z_p)$ blows up in finite time T, in particular,*
$$\lim_{t\to T} X_p(t) = +\infty \quad \lim_{t\to T} Z_p(t) = 0$$
and there exists $t_1 < T$ such that $X_p(t_1) = \tilde{N}_+ - 2$. ($X_p(t_1) = \tilde{N}_- - 2$ in the case of $\mathcal{M}^-_{\lambda,\Lambda}$)

Concerning the entire positive solution of (2.1), the classification, recalled in Definition 1.1, induces an analogous classification for the orbits of (2.5), which is the following one, (see [4, Section 3] for details and proofs).

Proposition 2.3. *If u_p is a regular solution of (2.1), positive in $(0, \infty)$, then for the corresponding trajectory Γ_p of (2.5) it holds:*

(i) *u_p is fast decaying $\iff \omega(\Gamma_p) = A_0$*
(ii) *u_p is slow decaying $\iff \omega(\Gamma_p) = M_0$*
(iii) *u_p is pseudo-slow decaying $\iff \omega(\Gamma_p)$ is a periodic orbit around M_0.*

Considering also the positive solutions of the Dirichlet problem in a ball, given by Proposition 2.2(ii), we define the sets:
$$\begin{aligned} \mathcal{C} = \big\{p > 1 : \text{ for } \Gamma_p = (X_p, Z_p) \text{ it holds that } \lim_{t\to T} X_p(t) = +\infty \\ \text{and } \lim_{t\to T} Z_p(t) = 0 \text{ for some } T > 0\big\} \end{aligned} \tag{2.13}$$
$$\mathcal{F} = \{p > 1 : \omega(\Gamma_p) = A_0\} \tag{2.14}$$
$$\mathcal{P} = \{p > 1 : \omega(\Gamma_p) \text{ is a periodic orbit around } M_0\} \tag{2.15}$$

$$\mathcal{S} = \{p > 1 : \omega(\Gamma_p) = M_0\} \tag{2.16}$$

3. Nonexistence of periodic orbits

3.1. $\mathcal{M}^+_{\lambda,\Lambda}$ **case.** Here we analyze the dynamical system (2.5) in the case of the operator $\mathcal{M}^+_{\lambda,\Lambda}$, with the aim of providing estimates on the values of p for which periodic orbits do not exist. We follow the approach of [4] which was used in [5] in the bi-dimensional case.

By [4, Proposition 2.10], we already know that there are no periodic orbits for $1 < p \le \frac{N+2}{N-2}$ or $p > \frac{\tilde N_+ +2}{\tilde N_+ -2}$. Moreover for $p = \frac{\tilde N_+ +2}{\tilde N_+ -2}$ we know that M_0 is a center which means that there exist infinitely many periodic orbits around M_0. Therefore we assume that

$$\frac{N+2}{N-2} < p < \frac{\tilde N_+ + 2}{\tilde N_+ - 2}\,. \tag{3.1}$$

Moreover, again by [4, Proposition 2.10], we know that for $p \ne \frac{\tilde N_+ +2}{\tilde N_+ -2}$ any periodic orbit $\mathcal{O}$ must intersect both R^+_λ and R^-_λ and cannot be tangent to the line ℓ_+. For a given periodic orbit $\mathcal{O}$ we denote by

$$P_1 = (x_1, \lambda(N-1)) \quad \text{and} \quad P_2 = (x_2, \lambda(N-1)), \quad x_1 < x_2, \tag{3.2}$$

the two points in the intersection $\mathcal{O} \cap \ell_+$ and by D the bounded region inside the orbit $\mathcal{O}$.

Both sets $D_1 = D \cap R^+_\lambda$ and $D_2 = D \cap R^-_\lambda$ have positive measure, and by the direction of the flow induced by (2.5), it is easy to see that

$$D_1 \subset Q_1 \quad \text{and} \quad D_2 \supset Q_2\,, \tag{3.3}$$

where $Q_1 = (x_1, x_2) \times (\lambda(N-1), \lambda N)$, $Q_2 = (x_1, x_2) \times (\Lambda(\tilde N_+ - 2), \lambda(N-1))$ (see Figure 2).

Then we define the function

$$\varphi(X, Z) = X^\alpha Z^\beta, \quad \text{with } \alpha = \frac{2}{p-1}, \quad \beta = \frac{3-p}{p-1}, \tag{3.4}$$

and denote

$$a' = \frac{\int_{D_1} \varphi \, dX\, dZ}{\int_{D_2} \varphi \, dX\, dZ}\,, \quad a'' = \frac{\int_{Q_1} \varphi \, dX\, dZ}{\int_{Q_2} \varphi \, dX\, dZ}\,. \tag{3.5}$$

Finally we consider the families of subsets

$$\begin{aligned} \Sigma_1 &= \{S \subset \mathbb{R}^2 : |S| > 0,\, D_1 \subset \bar S \subseteq Q_1\}, \\ \Sigma_2 &= \{S \subset \mathbb{R}^2 : |S| > 0,\, Q_2 \subseteq \bar S \subset D_2\}\,. \end{aligned}$$

Obviously, for any couple of sets $(S_1, S_2) \in \Sigma_1 \times \Sigma_2$ we have that the number

$$a = a(S_1, S_2) = \frac{\int_{S_1} \varphi \, dX\, dZ}{\int_{S_2} \varphi \, dX\, dZ} \tag{3.6}$$

belongs to the interval $(a', a'']$.

We have the following condition for the existence of a periodic orbit.

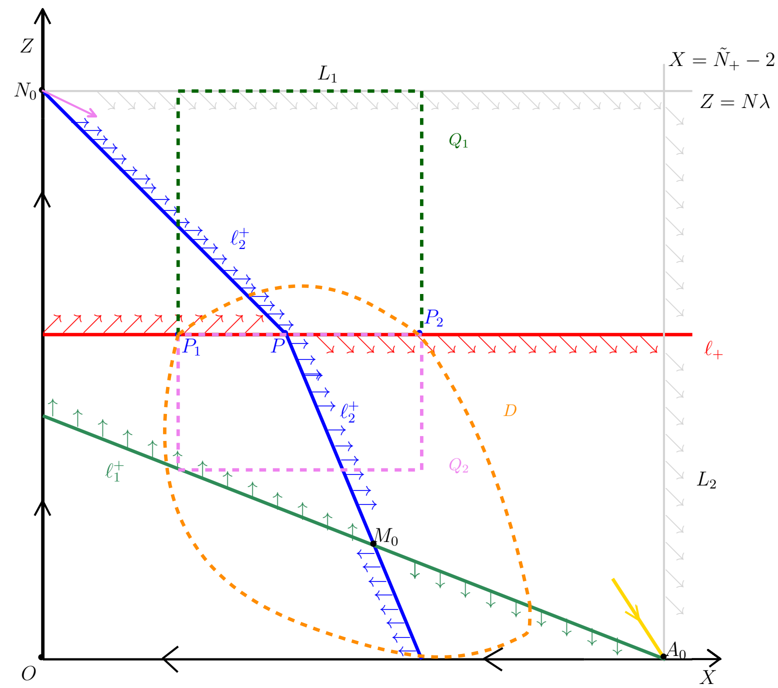

FIGURE 2. Periodic orbit and the sets Q_1, Q_2

Proposition 3.1. *Let $p \in \left(\frac{N+2}{N-2}, \frac{\tilde{N}_++2}{\tilde{N}_+-2}\right)$ and assume a periodic orbit $\mathcal{O}_p$ for* (2.5) *exists. Then, for every $a = a(S1, S2)$ as in* (3.6) *it holds*

$$p > h(a), \quad h(a) = \frac{a(N+2) + (\tilde{N}_+ + 2)}{a(N-2) + (\tilde{N}_+ - 2)}, \quad h(a) \in \Big(\frac{N+2}{N-2}, \frac{\tilde{N}_+ + 2}{\tilde{N}_+ - 2}\Big). \tag{3.7}$$

Proof. Since p satisfies (3.1), it follows that

$$-p(\tilde{N}_+ - 2) > -\tilde{N}_+ - 2 \quad \text{and} \quad -p(N-2) < -N-2 \tag{3.8}$$

We consider the function φ as in (3.5) and define

$$\Phi(X, Z) = \partial_X(\varphi f) + \partial_Z(\varphi g) \tag{3.9}$$

with f and g as in (2.6a),(2.6b). By computations we obtain

$$\Phi(X, Z) = \begin{cases} X^\alpha Z^\beta \frac{-p(N-2)+(N+2)}{p-1} & \text{if } (X, Z) \in R_\lambda^+ \\ X^\alpha Z^\beta \frac{-p(\tilde{N}_+-2)+(\tilde{N}_++2)}{p-1} & \text{if } (X, Z) \in R_\lambda^- . \end{cases} \tag{3.10}$$

Then, using the above notation we have

$$0 = \int_{\partial D} \varphi\{f dZ - g dX\} = \int_D \Phi \, dX \, dZ = \int_{D_1} \Phi \, dX \, dZ + \int_{D_2} \Phi \, dX \, dZ \,,$$

where the first equality holds because $dX = f dt$ and $dZ = g dt$. Hence, by (3.10),

$$0 = \frac{-p(N-2) + (N+2)}{p-1} \int_{D_1} X^\alpha Z^\beta \, dX \, dZ$$

$$+\frac{-p(\tilde N_+-2)+(\tilde N_++2)}{p-1}\int_{D_2}X^\alpha Z^\beta\,dX\,dZ.$$

By (3.8) the coefficient of the first term is negative, while the second one is positive. Hence, since $\varphi>0$, we obtain that for every $(S_1,S_2)\in\Sigma_1\times\Sigma_2$:

$$0>\frac{-p(N-2)+(N+2)}{p-1}\int_{S_1}\varphi\,dX\,dZ+\frac{-p(\tilde N_+-2)+(\tilde N_++2)}{p-1}\int_{S_2}\varphi\,dX\,dZ\,.$$

Thus for any $a=a(S_1,S_2)$ defined as in (3.6) we have

$$0>\frac{-p(N-2)a-p(\tilde N_+-2)+(N+2)a+(\tilde N_++2)}{p-1}\int_{S_2}\varphi\,dX\,dZ\,.$$

Since $\varphi>0$, the above inequality implies

$$\frac{-p(N-2)a-p(\tilde N_+-2)+(N+2)a+(\tilde N_++2)}{p-1}<0$$

which is equivalent to (3.7). Finally a straightforward computation gives that $h(a)$ belongs to the interval $(\frac{N+2}{N-2},\frac{\tilde N_++2}{\tilde N_+-2})$. $\square$

Note that the function $h(a)$ in (3.7) is decreasing with respect to a.

The above result will be used to obtain a bound for the values of p for which the corresponding dynamical system (2.5) cannot admit periodic orbits. This can be achieved by taking a couple of sets $(S_1,S_2)\in\Sigma_1\times\Sigma_2$ such that the number $a=a(S_1,S_2)$ can be estimated by a constant which depends only on the dimension N and the ellipticity constants λ and Λ. This is the content of the next two corollaries in which we choose particular sets (S_1,S_2).

Corollary 3.2. *Let*

$$\hat a_+=\frac{(\frac{N}{N-1})^{\frac{N-2}{2}}-1}{1-(\frac{\Lambda(\tilde N_+-2)}{\lambda(N-1)})^{\frac{\tilde N_+-2}{2}}}. \tag{3.11}$$

Then for

$$p\le h(\hat a_+)=\frac{\hat a_+(N+2)+(\tilde N_++2)}{\hat a_+(N-2)+(\tilde N_+-2)} \tag{3.12}$$

system (2.5) *does not admit periodic orbits.*

Proof. Let us assume that (3.12) holds and that there exists a periodic orbit for (2.5). Then we choose the couple of rectangles (Q_1,Q_2) defined as in (3.3) to compute and estimate $a''=a(Q_1,Q_2)$ (see (3.6)). Integrating we obtain

$$\begin{aligned}\int_{Q_1}\varphi\,dX\,dZ&=\int_{Q_1}X^\alpha Z^\beta\,dX\,dZ\\&=\frac{(\lambda(N))^{\beta+1}-(\lambda(N-1))^{\beta+1}}{\beta+1}\int_{x_1}^{x_2}X^\alpha dX\end{aligned} \tag{3.13}$$

and analogously for Q_2,

$$\begin{aligned}\int_{Q_2}\varphi\,dX\,dZ&=\int_{Q_2}X^\alpha Z^\beta\,dX\,dZ\\&=\frac{(\lambda(N-1))^{\beta+1}-(\Lambda(\tilde N_+-2))^{\beta+1}}{\beta+1}\int_{x_1}^{x_2}X^\alpha dX.\end{aligned} \tag{3.14}$$

Thus we obtain

$$a'' = a(Q_1, Q_2) = \frac{(\frac{N}{N-1})^{\beta+1} - 1}{1 - (\frac{\Lambda(\tilde{N}_+-2)}{\lambda(N-1)})^{\beta+1}} < \frac{(\frac{N}{N-1})^{\frac{N-2}{2}} - 1}{1 - (\frac{\Lambda(\tilde{N}_+-2)}{\lambda(N-1)})^{\frac{\tilde{N}_+-2}{2}}} = \hat{a}_+$$

Since the function $h(a)$ is decreasing, by Proposition 3.1 we deduce that

$$p > h(a'') > h(\hat{a}_+)$$

which contradicts assumption (3.12). □

To improve the estimate (3.12) it would be better to take a couple (S_1, S_2) with S_1 contained in Q_1 and S_2 containing Q_2. For the next result we consider the couple $(Q_1, Q_2 \cup T)$ where

$$T = \mathrm{co}(A_1, A_2, M_0)$$

which is the convex hull of the points

$$A_1 = (x_1, \Lambda(\tilde{N}_+ - 2)), \quad A_2 = (x_2, \Lambda(\tilde{N}_+ - 2)), \quad M_0\,,$$

where M_0 is the stationary point of Proposition 2.1 (see Figure 3).

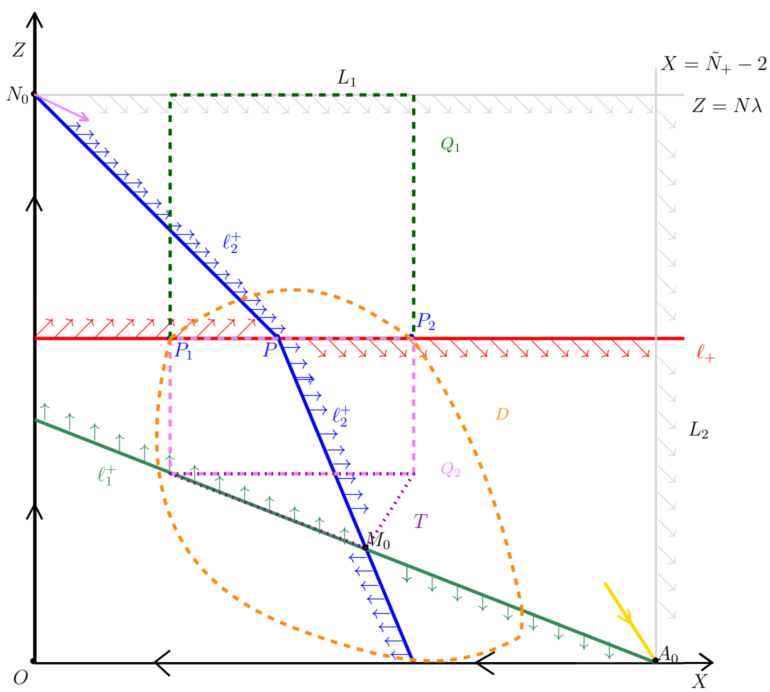

Figure 3. Periodic orbit and relevant sets

Corollary 3.3. *Let*

$$\tilde{a}_+ = \tilde{a}_+(p)$$

$$= \frac{\frac{(\lambda(N))^{\beta+1} - (\lambda(N-1))^{\beta+1}}{\beta+1}}{\frac{(\lambda(N-1))^{\beta+1} - (\Lambda(\tilde{N}_+-2))^{\beta+1}}{\beta+1} + \frac{\Lambda}{p-1}\min\{(\Lambda(\tilde{N}_+ - 2 - \frac{2}{p-1})^{\beta}, (\Lambda(\tilde{N}_+ - 2)^{\beta}\}}. \tag{3.15}$$

Then for

$$p \le h(\tilde a_+) = \frac{\tilde a_+(N+2) + (\tilde N_+ + 2)}{\tilde a_+(N-2) + (\tilde N_+ - 2)} \tag{3.16}$$

system (2.5) *does not admit any periodic orbit.*

Proof. Integrating the function $\varphi(X,Z) = X^\alpha Z^\beta$ on the triangle T we obtain

$$\begin{aligned}\int_T X^\alpha Z^\beta \, dX\, dZ &\ge \inf_T Z^\beta \int_T X^\alpha \, dX\, dZ\\ &= \frac{1}{2}\int_{Q_3} X^\alpha \, dX\, dZ \inf_T Z^\beta\\ &= \int_{x_1}^{x_2} X^\alpha dX \frac{\Lambda}{p-1} \inf_T Z^\beta\end{aligned}$$

since $\int_T X^\alpha \, dX\, dZ$ is half of the integral on the rectangle

$$Q_3 = (x_1, x_2) \times \Big(\Lambda(\tilde N_+ - 2 - \frac{2}{p-1}), \Lambda(\tilde N_+ - 2)\Big).$$

Using (3.13) and (3.14) we obtain

$$\begin{aligned}&a(Q_1, Q_2 \cup T) \le \tilde a_+\\ &= \frac{\frac{(\lambda(N))^{\beta+1} - (\lambda(N-1))^{\beta+1}}{\beta+1}}{\frac{(\lambda(N-1))^{\beta+1} - (\Lambda(\tilde N_+ - 2))^{\beta+1}}{\beta+1} + \frac{\Lambda}{p-1}\min\{(\Lambda(\tilde N_+ - 2 - \frac{2}{p-1})^\beta, (\Lambda(\tilde N_+ - 2)^\beta\}}\end{aligned}$$

Since the function h is decreasing, we deduce from Proposition 3.1 that if a periodic orbit exists then

$$p > h(a(Q_1, Q_2 \cup T)) \ge h(\tilde a_+).$$

Hence we obtain the assertion by the assumption (3.16). $\square$

Note that while the right hand side of (3.12) does not depend on p, in the estimate (3.16) we have that $\tilde a_+$ depends on p, so that inequality (3.16) is more difficult to use to have explicit bounds on p. Nevertheless we will use it in Section 5 in some particular cases to show sharper estimates.

3.2. $\mathcal{M}^-_{\lambda,\Lambda}$ **case.** Here we analyze the dynamical system (2.5) in the case of the operator $\mathcal{M}^-_{\lambda,\Lambda}$. Though the statements and proofs are similar to those for $\mathcal{M}^+_{\lambda,\Lambda}$, we repeat the main steps for the reader's convenience. Based on the result of [4, Proposition 2.10], we restrict our attention only to a certain interval of values of p in order to obtain better estimates for the critical exponent, hence we assume

$$\frac{\tilde N_- + 2}{\tilde N_- - 2} < p < \frac{N+2}{N-2}\,. \tag{3.17}$$

Moreover, again by [4, Proposition 2.10], we know that for $p \ne \frac{N+2}{N-2}$ any periodic orbit $\mathcal{O}$ must intersect both R^+_Λ and R^-_Λ and cannot be tangent to the line ℓ_-. For a given periodic orbit $\mathcal{O}$ we denote by:

$$P_1 = (x_1, \Lambda(N-1)) \quad \text{and} \quad P_2 = (x_2, \Lambda(N-1))\,, \quad x_1 < x_2 \tag{3.18}$$

the two points in the intersection $\mathcal{O} \cap \ell_-$ and by D the bounded region inside the orbit $\mathcal{O}$.

The sets $D_1 = D \cap R_\Lambda^+$ and $D_2 = D \cap R_\Lambda^-$ have both positive measure and, by the direction of the flow induced by (2.5), it is easy to see that

$$D_1 \subset Q_1 \quad \text{and} \quad D_2 \supset Q_2\,, \tag{3.19}$$

where $Q_1 = (x_1, x_2) \times (\Lambda(N-1), \Lambda N)$, $Q_2 = (x_1, x_2) \times (\lambda(\tilde{N}_- - 2), \Lambda(N-1))$ (see Figure 4).

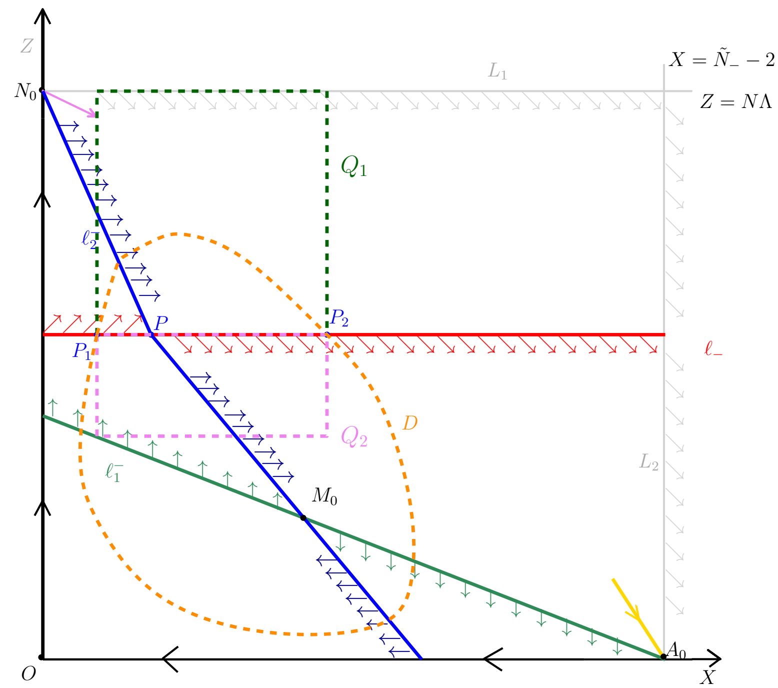

FIGURE 4. Periodic orbit and auxiliary sets

Now we recall the definition of the function φ defined in (3.5)

$$\varphi(X, Z) = X^\alpha Z^\beta, \quad \text{with } \alpha = \frac{2}{p-1}\,,\ \beta = \frac{3-p}{p-1} \tag{3.20}$$

and as done in the $\mathcal{M}^+_{\lambda,\Lambda}$ case, we denote

$$a' = \frac{\int_{D_1} \varphi\, dX\, dZ}{\int_{D_2} \varphi\, dX\, dZ}\,, \quad a'' = \frac{\int_{Q_1} \varphi\, dX\, dZ}{\int_{Q_2} \varphi\, dX\, dZ}, \tag{3.21}$$

and the families of subsets:

$$\Sigma_1 = \{S \subset \mathbb{R}^2 : |S| > 0\,,\ D_1 \subset \bar{S} \subseteq Q_1\},$$
$$\Sigma_2 = \{S \subset \mathbb{R}^2 : |S| > 0\,,\ Q_2 \subseteq \bar{S} \subset D_2\}$$

Obviously, for any couple of sets $(S_1, S_2) \in \Sigma_1 \times \Sigma_2$ we have that the number

$$a = a(S_1, S_2) = \frac{\int_{S_1} \varphi\, dX\, dZ}{\int_{S_2} \varphi\, dX\, dZ} \tag{3.22}$$

belongs to the interval $(a', a'']$.

We have the following condition for the existence of a periodic orbit.

Proposition 3.4. *Let $p \in \big(\frac{\tilde N_-+2}{\tilde N_--2}, \frac{N+2}{N-2}\big)$ and assume a periodic orbit $\mathcal{O}_p$ for (2.5) exists. Then, for every $a = a(S1, S2)$ as in (3.22) it holds*

$$p < h(a), \quad h(a) = \frac{a(N+2) + (\tilde N_- + 2)}{a(N-2) + (\tilde N_- - 2)}, \quad h(a) \in \Big(\frac{\tilde N_- + 2}{\tilde N_- - 2}, \frac{N+2}{N-2}\Big). \tag{3.23}$$

Proof. The proof is similar to the one for the operator $\mathcal{M}^+_{\lambda,\Lambda}$, given in Proposition 3.1. We write the computations for the reader's convenience. Since p satisfies (3.17),

$$-p(\tilde N_- - 2) < -\tilde N_- - 2 \quad \text{and} \quad -p(N-2) > -N-2\,. \tag{3.24}$$

Again as done in the $\mathcal{M}^+_{\lambda,\Lambda}$ case we define

$$\Phi(X,Z) = \partial_X(\varphi f) + \partial_Z(\varphi g)\,, \tag{3.25}$$

with φ as in (3.5), f and g as in (2.6a),(2.6b). By computations we obtain

$$\Phi(X,Z) = \begin{cases} X^\alpha Z^\beta \frac{-p(N-2)+(N+2)}{p-1} & \text{if } (X,Z) \in R^+ \\ X^\alpha Z^\beta \frac{-p(\tilde N_+-2)+(\tilde N_++2)}{p-1} & \text{if } (X,Z) \in R^-\,. \end{cases} \tag{3.26}$$

It follows from the above computation that

$$0 = \int_{\partial D} \varphi\{f dZ - g dX\} = \int_D \Phi\, dX\, dZ = \int_{D_1} \Phi\, dX\, dZ + \int_{D_2} \Phi\, dX\, dZ\,,$$

where the first equality holds because $dX = f dt$ and $dZ = g dt$. Hence, by (3.26),

$$\begin{aligned} 0 = & \frac{-p(N-2) + (N+2)}{p-1} \int_{D_1} X^\alpha Z^\beta\, dX\, dZ \\ & + \frac{-p(\tilde N_- - 2) + (\tilde N_- + 2)}{p-1} \int_{D_2} X^\alpha Z^\beta\, dX\, dZ. \end{aligned}$$

By (3.24) the first right term coefficient is positive, while the second one is negative. Hence, since $\varphi > 0$, we obtain that for every $(S_1, S_2) \in \Sigma_1 \times \Sigma_2$,

$$0 < \frac{-p(N-2) + (N+2)}{p-1} \int_{S_1} \varphi\, dX\, dZ + \frac{-p(\tilde N_- - 2) + (\tilde N_- + 2)}{p-1} \int_{S_2} \varphi\, dX\, dZ\,.$$

Thus for any $a = a(S_1, S_2)$ defined as in (3.6) we have

$$0 < \frac{-p(N-2)a - p(\tilde N_- - 2) + (N+2)a + (\tilde N_- + 2)}{p-1} \int_{S_2} \varphi\, dX\, dZ\,.$$

Since $\varphi > 0$, the above inequality implies

$$\frac{-p(N-2)a - p(\tilde N_- - 2) + (N+2)a + (\tilde N_- + 2)}{p-1} > 0$$

which is equivalent to (3.23). Finally a straightforward computation gives that $h(a)$ belongs to the interval $(\frac{\tilde N_-+2}{\tilde N_--2}, \frac{N+2}{N-2})$. $\square$

Note that the function $h(a)$ in (3.23) is increasing with respect to a. Taking particular couples of sets in $\Sigma_1 \times \Sigma_2$, as in the case of $\mathcal{M}^+_{\lambda,\Lambda}$, we obtain the following results.

Corollary 3.5. *Let*

$$\hat{a}_- = \frac{\left(\frac{N}{N-1}\right)^{\frac{N-2}{2}} - 1}{1 - \left(\frac{\lambda(\tilde{N}_- -2)}{\Lambda(N-1)}\right)^{\frac{\tilde{N}_+ -2}{2}}}\,. \tag{3.27}$$

Then for

$$p \geq h(\hat{a}_-) = \frac{\hat{a}_-(N+2) + (\tilde{N}_- + 2)}{\hat{a}_-(N-2) + (\tilde{N}_- - 2)} \tag{3.28}$$

system (2.5) *does not admit periodic orbits.*

Proof. It is similar to the one of Corollary 3.2 choosing the couple of rectangles (Q_1, Q_2) defined in (3.19) to compute and estimate$a'' = a(Q_1, Q_2)$. ☐

To improve the estimate (3.23) we consider the couple $(Q_1, Q_2 \cup T)$ where

$$T = \mathrm{co}(A_1, A_2, M_0)$$

which is the convex hull of the points

$$A_1 = (x_1, \lambda(\tilde{N}_- - 2)), \quad A_2 = (x_2, \lambda(\tilde{N}_- - 2)), \quad M_0,$$

where M_0 is the stationary point of Proposition 2.1 (see Figure 5).

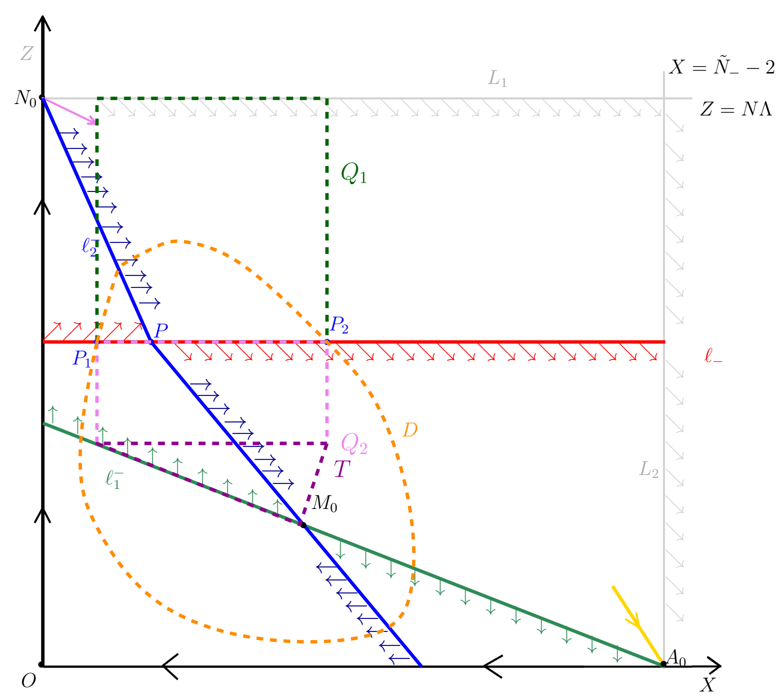

Figure 5. Auxiliary sets

Analogously to Corollary 3.3 and with a similar proof obtain the following result.

Corollary 3.6. *Let*

$$\begin{aligned}\tilde a_- &= \tilde a_-(p)\\ &= \frac{\frac{(\Lambda(N))^{\beta+1}-(\Lambda(N-1))^{\beta+1}}{\beta+1}}{\frac{(\Lambda(N-1))^{\beta+1}-(\lambda(\tilde N_- -2))^{\beta+1}}{\beta+1}+\frac{\lambda}{p-1}\min\{(\lambda(\tilde N_- - 2-\frac{2}{p-1})^{\beta},(\lambda(\tilde N_- -2)^{\beta}\}}.\end{aligned} \tag{3.29}$$

Then for

$$p \ge h(\tilde a_-) = \frac{\tilde a_-(N+2)+(\tilde N_- +2)}{\tilde a_-(N-2)+(\tilde N_- -2)} \tag{3.30}$$

system (2.5) *does not admit any periodic orbit.*

4. Pseudo-slow decaying solutions and critical exponents

We first consider the operator $\mathcal{M}^+_{\lambda,\Lambda}$. To prove Theorem 1.3, we start by showing the connection between the critical exponent p^*_+ and the existence of periodic orbits.

As recalled in the introduction the critical exponent p^*_+ is defined by the propriety of being the only exponent p for which (1.1) admits an entire fast decaying solution. In [4] it is proved that this is equivalent to define

$$p^*_+ = \sup \mathcal{C}\,, \tag{4.1}$$

where the set $\mathcal{C}$ is defined in (2.13). The following result is important for deriving the estimate for p^*_+.

Proposition 4.1. *Let $p \in (\frac{N+2}{N-2}, \frac{\tilde N_+ +2}{\tilde N_+ -2})$ and assume that no periodic orbits exist for the system* (2.5) *(corresponding to such p). Then $p \in \mathcal{C}$.*

Proof. Considering the classification of the sets $\mathcal{C}$, $\mathcal{F}$, $\mathcal{P}$, $\mathcal{S}$, given in (2.13)-(2.16), we have that obviously $p \notin \mathcal{P}$. Moreover, since M_0 is a source for $p < \frac{\tilde N_+ +2}{\tilde N_+ -2}$, then $p \notin \mathcal{S}$.

Finally if p would belong to $\mathcal{F}$, then the regular trajectory Γ_p, together with the X and Z axis, would enclose an invariant set for the flow which would not allow any ω-limit for the trajectories starting from M_0. Thus, $p \in \mathcal{C}$ (see Figure 6). □

Proof of Theorem 1.3. The first assertion follows by Corollary 3.2 because if there are no periodic orbits for the dynamical system (2.5) there cannot be pseudo-slow decaying solutions for (1.1).

The estimate (1.5) on the critical exponent p^*_+ is a consequence of (4.1), Proposition 4.1, and Corollary 3.2. □

One could get a sharper bound for the critical exponent by considering the number

$$h(\tilde a_+) = \frac{\tilde a_+(N+2)+(\tilde N_+ +2)}{\tilde a_+(N-2)+(\tilde N_+ -2)} \tag{4.2}$$

for $\tilde a_+ = \tilde a_+(p)$ defined in (3.16) and applying Corollary 3.3

Now we study the case of the operator $\mathcal{M}^-_{\lambda,\Lambda}$. To prove Theorem 1.4 we use the following result, where $h(\hat a_-)$ is defined as in (3.27).

Proposition 4.2. *If $p \ge h(\hat a_-)$, then $p \notin \mathcal{C}$.*

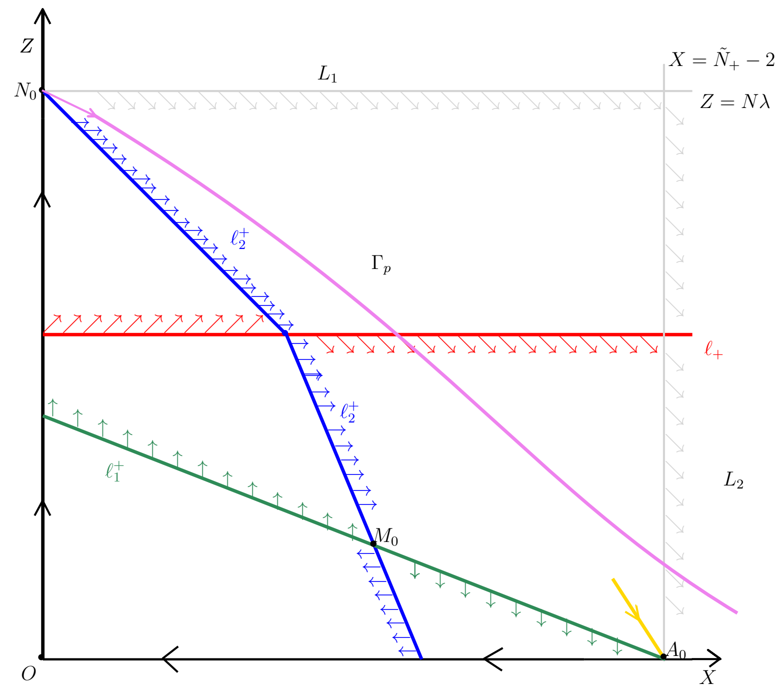

Figure 6. The case when there are no periodic orbits

Proof. Since $p \geq h(\hat{a}_-) > \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2}$, M_0 is a sink while N_0 and A_0 are saddle points. Moreover there are no periodic orbits by Corollary 3.5.

Suppose $p \in \mathcal{C}$, then the trajectory Γ_p starting from N_0 blows up forwardly in finite time after crossing the line $L_1 = \{(X, Z) : X = \tilde{N}_- - 2\}$. This implies that the domain bounded by the X and Z axes, the orbit Γ_p and the line L_1 is such that any orbit inside it can only exit in forward time (through L_1), but not in backward time. Thus, the unique trajectory τ_p whose ω-limit is A_0 cannot go anywhere backward in time and this provides a contradiction (see Figure 7). □

Proof of Theorem 1.4. It is similar to that of Theorem 1.3 using Corollary 3.5, Proposition 4.2 and the fact that

$$p_-^* = \sup \mathcal{C}$$

as shown in [4]. □

As in the case for $\mathcal{M}^+_{\lambda,\Lambda}$, we can obtain a sharper estimate by considering the number

$$h(\tilde{a}_-) = \frac{\tilde{a}_-(N+2) + (\tilde{N}_- + 2)}{\tilde{a}_-(N-2) + (\tilde{N}_- - 2)} \tag{4.3}$$

for $\tilde{a}_-$ defined in (3.30) and applying Corollary 3.6.

5. Concluding remarks

To better appreciate the newer bound for both operators we would like to present a few examples of how they compare to the one previously known. To this aim let

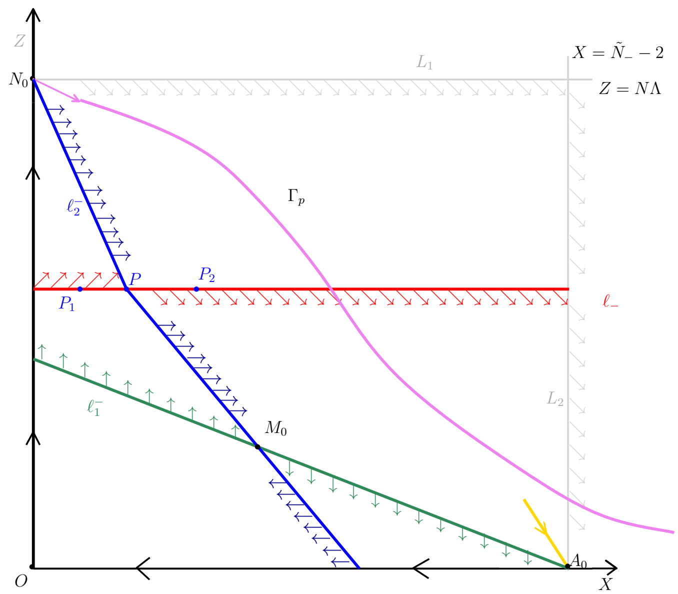

FIGURE 7. Contradiction arising in Proposition 4.2

us define

$$\tilde{p}_+ = \sup\{p \text{ solutions of } (3.16)\}, \tag{5.1}$$

$$\tilde{p}_- = \inf\{p \text{ solutions of } (3.30)\}. \tag{5.2}$$

Note that $\tilde{a}_+$ and $\tilde{a}_-$ depend on p. Then, based on what was shown in Section 4 we obtain the following ordering.

For $\mathcal{M}^+_{\lambda,\Lambda}$:

$$\frac{\tilde{N}_+ + 2}{\tilde{N}_+ - 2} > p^*_+ > \tilde{p}_+ > h(\hat{a}_+) > \frac{N+2}{N-2}\,,$$

where $\tilde{p}_+$ and $h(\hat{a}_+)$ are defined in (5.1) and (3.12) respectively, while the other bounds are from [2] and Proposition 3.1.

For $\mathcal{M}^-_{\lambda,\Lambda}$:

$$\frac{\tilde{N}_+ + 2}{\tilde{N}_+ - 2} < p^*_- < \tilde{p}_- < h(\hat{a}_-) < \frac{N+2}{N-2}\,,$$

where $\tilde{p}_-$ and $h(\hat{a}_-)$ are defined in (5.2) and (3.28) respectively.

In the case of $\mathcal{M}^+_{\lambda,\Lambda}$ we can appreciate the difference by computing each one of the bounds in low dimensions. For that we will fix $\lambda = 1, \Lambda = 2$ and vary N from 4 to 8. Then we can observe the values of $\tilde{p}_+$,$h(\hat{a}_+)$ and $\frac{N+2}{N-2}$ on the following table

Another interesting case is when we consider a class of problems driven by $\mathcal{M}^+_{\lambda,N-1}$ (i.e. $\Lambda = N-1$) in $\mathbb{R}^N$ for $\lambda > 1$. It follows from [2] that for every dimension N greater than 2, the critical exponent p^*_+ is bounded uniformly from above by the constant $\frac{\tilde{N}_+ + 2}{\tilde{N}_+ - 2} = \frac{\lambda+3}{\lambda-1}$.

On the other hand, as we may observe from the graph in Figure 8, while the classical lower bound presented in [2], i.e. $\frac{N+2}{N-2}$ converges to 1 as N goes to $+\infty$,

TABLE 1.

N	$\frac{N+2}{N-2}$	$h(\hat{a}_+)$	$\tilde{p}_+$	$\frac{\tilde{N}_++2}{\tilde{N}_+-2}$
4	3	3.92	6.47	9
5	2.3	2.85	3.93	5
6	2	2.36	3	3.67
7	1.8	2.07	2.44	3
8	1.67	1.88	2.11	2.6

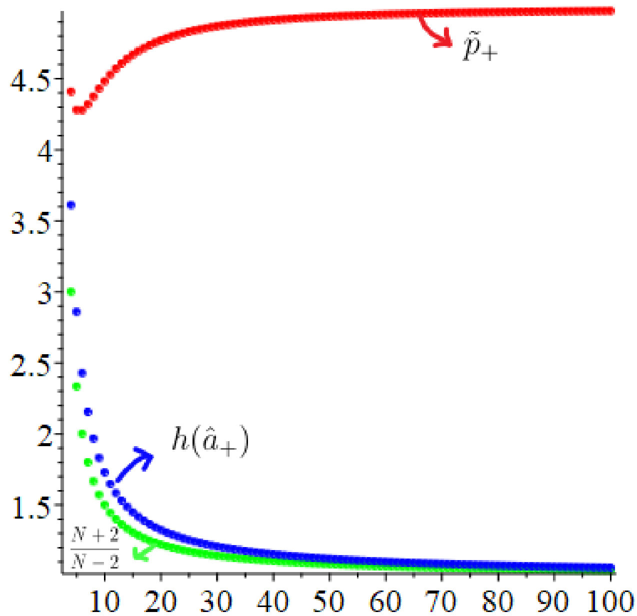

FIGURE 8. Graphs of $\tilde{p}_+$, $h(\hat{a}_+)$ and $\frac{N+2}{N-2}$, for $\lambda = 2, \Lambda = N - 1$

the bound obtained in this paper by considering more refined set, namely the one obtained in Corollary 3.3, converges to $\frac{\lambda+3}{\lambda-1}$ by (4.2), since by (3.15):

$$\lim_{N\to+\infty} \tilde{a}_+ = 0$$

Hence this proves that asymptotically the critical exponent of the problem driven by $\mathcal{M}^+_{\lambda,N-1}$ in $\mathbb{R}^N$ converges to $\frac{\lambda+3}{\lambda-1} = \frac{\tilde{N}_++2}{\tilde{N}_+-2}$.

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Filomena Pacella
Dipartimento di Matematica, Universityà di Roma Sapienza, Italy
Email address: pacella@mat.uniroma1.it

David Stolnicki
Dipartimento di Matematica, Universityà di Roma Sapienza, Italy
Email address: david.stolnicki@uniroma1.it