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SPECTRAL THEORY OF $\mathcal{C}$-SYMMETRIC NON-SELFADJOINT DIFFERENTIAL OPERATORS OF ORDER $2n$

HORST BEHNCKE, DON HINTON

Abstract. We continue the spectral analysis of differential operators with complex coefficients, extending some results for Sturm-Liouville operators to higher order operators. We give conditions for the essential spectrum to be empty, and for the operator to have compact resolvent. Conditions are given on the coefficients for the resolvent to be Hilbert-Schmidt. These conditions are new even for real coefficients, i.e., the selfadjoint case. Asymptotic analysis is a central tool.

1. Introduction

Non-selfadjoint operators (NSA) arise in many areas of theoretical physics. Yet, compared to thousands of papers on selfadjoint operators arising in differential equations, there are fewer papers on their NSA counterparts. This is so mainly because of the absence of the spectral theorem and the order properties of the real numbers. Thus there are no Sturm theorems and no spectral representations. Special tools like subordinacy and the m-matrix as the Borel transforms of the spectral measure are likewise missing. This is not surprising and it can be already seen for Sturm-Liouvillve operators. New phenomena arise, like empty spectrum, all of $\mathbb{C}$ as spectrum, or higher order poles for the resolvent. Hence the analysis the NSA differential operators so far lacks order and guiding principles.

Most studies rely on numerical range conditions [9, 10] which led among others the analysis of dissipative operators. The Russian school, Naimark, Ljance, et al. [30] in their study of NSA Sturm-Liouville operators use function theoretic methods and Fourier transforms in their analyses of the eigenfunction expansion of Sturm-Liouville NSA operators. Yet this approach came to a stop in the mid 1970's. Sims [38] was able to extend Weyl's program by construction of the m-function for $-y'' + q(x)y$ where q is complex valued with $\operatorname{Im} q(x) \leq 0$. This was extended by Brown, McCormack, Evans, and Plum [9] to general Sturm-Liouville operators and later to complex Hamiltonians [10, 29]. All these studies rely heavily on numerical range conditions to localize the spectrum and construct the m-matrix by further assumptions on the numerical range. However, unlike the real case, the meaning

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of the m-matrix remains obscure, even for Sturm-Liouville constant coefficients operators.

This article is devoted to the spectral analysis of differential operators of the form

$$T[y] = \frac{1}{w} \sum_{k=0}^{n} (-1)^k (p_k y^{(k)})^{(k)} \quad \text{on } \mathcal{L}_w^2(I). \tag{1.1}$$

Here I is an interval in $\mathbb{R}$, and w is the weight function which defines the scalar product. Mostly we will be dealing with the one singular endpoint case $I = [a, \infty)$. The coefficients p_k are assumed to fulfill the usual requirements, i.e., $1/p_n$, p_k for $k = 0, \dots, n-1$, are locally integrable and complex valued. The maximal operator $T_{\max}$ has domain $D(T_{\max})$ consisting of all functions y in $\mathcal{L}_w^2(I)$ to which T can be applied and have $T[y] \in \mathcal{L}_w^2(I)$. $D(T_{\min})$, the domain of the minimal operator, is the closure of the set of all functions in $D(T_{\max})$ with compact support in the interior of I. Both $T_{\min}$ and $T_{\max}$ are closed, densely defined operators. For the systems formulation of (1.1) below in (2.7), we define the quasi-derivatives $y^{[k]}$ by [2]

$$\begin{gathered} y^{[k]} = y^{(k)}, \quad 0 \le k \le n-1, \quad y^{[n]} = p_n y^{(n)}, \\ y^{[n+k]} = -y^{[n+k-1]'} + p_{n-k} y^{[n-k]}, \end{gathered} \tag{1.2}$$

for $1 \le k \le n-1$, in which case

$$T[y] = \frac{1}{w} \big[y^{[2n-1]'} + p_0 y \big].$$

For $y \in D(T_{\min})$, $I = [a, \infty)$, it is known that $y^{[k]}(a) = 0$, $k = 0, \dots, 2n-1$, and from this it follows that $T_{\min}$ has no eigenvalues as the existence-uniqueness theory for (1.1) implies $y \equiv 0$ if $y \in D(T_{\min})$ and $T_{\min}[y] = zy$.

The formal adjoint T^+ of T is then given by

$$T^+[y] = \frac{1}{w} \sum_{k=0}^{n} (\bar{p}_k y^{(k)})^{(k)} \quad \text{on } \mathcal{L}_w^2(I). \tag{1.3}$$

We have the adjoint relations e.g., Goldberg [18, p. 130] or Kauffman, Read, and Zettl [22, p. 14],

$$T_{\min}^* = T_{\max}^+, \quad T_{\max} = T_{\min}^{+*}, \quad T_{\min} = T_{\max}^{+*}, \quad T_{\max}^* = T_{\min}^+,$$

where * is the Hilbert space adjoint. Further we have $T_{\min}$ is $\mathcal{C}-$symmetric, i.e., $T_{\min} \subset \mathcal{C} T_{\max}^+ \mathcal{C}$ where $\mathcal{C}$ is complex conjugation. In the case of $\mathcal{C}$-symmetric operators like $T_{\min}$, we are interested in the spectral theory of the $\mathcal{C}$-selfadjoint extensions T_s of $T_{\min}$, i.e., operators T_s which satisfy the relations

$$T_{\min} \subset T_s \subset T_{\max}, \quad T_s^* = \mathcal{C} T_s \mathcal{C}. \tag{1.4}$$

In general a $\mathcal{C}-$symmetric map $\mathcal{C}$ on a Hilbert space is one that is conjugate linear, involutive, and isometric. Our $\mathcal{C}$-symmetry is usually called $\mathcal{J}$-symmetry for complex conjugation.

These operators are obtained by imposing appropriate boundary conditions on the elements of $D(T_{\max})$, see Knowles [25]. In [8] the spectral analysis of such maximal $\mathcal{C}$-symmetric operators was based on asymptotic integration, because a good knowledge of the eigenfunctions allows to deduce spectral properties of T. Earlier asymptotic integration for differential equations with complex coefficients was mainly used for computing the deficiency index. References for this approach may be found in the book of Eastham [12] which is entirely devoted to asymptotic integration.

In fact we will follow the presentations of [6, 7, 8] closely so that we may cut the first three sections rather short. The spectrum, resolvent, the domain, null space, and the range of an operator T will be denoted by $\sigma(T)\rho(T)$, $D(T)$, $N(T)$, and $R(T)$ respectively. The numerical range of T is defined by

$$\mathcal{N}(T) = \{\langle Tx, x\rangle : x \in D(T), \|x\| = 1\}. \tag{1.5}$$

The set $\mathcal{N}(T)$ is convex, but need not be closed. $\overline{\mathcal{N}(T)}$ is an important tool in finding the spectrum - see section 2. Clearly, eigenvalues are contained in $\mathcal{N}(T)$. Let

$$\mathcal{N}_n(T) = \{\langle Tx, x\rangle : x \in D(T), \|x\| = 1, \text{ support } x \subseteq [n.\infty)\}.$$

Since many properties of T depend only on the asymptotics of the eigenfunctions, we define the essential numerical range of T by

$$\mathcal{N}_\infty(T) = \cap\overline{\mathcal{N}_n(T)}. \tag{1.6}$$

Note that the numerical range for $T_{\min}$ can be computed from

$$\langle T_{\min} y, y\rangle = \int_a^\infty \sum_{k=0}^n p_k(x)|y_k^{(k)}(x)|^2\, w(x)dx. \tag{1.7}$$

Thus the numerical range of $T_{\min}$ will lie in a sector of $\mathbb{C}$ of angle $\le \pi$ if all values of the coefficients lie in this sector.

$\mathfrak{L}_0^p = \mathfrak{L}_0^p([a,\infty))$, $1 \le p < \infty$, will denote the set of all p-integrable functions vanishing at infinity. For functions f and g we write $f \ll g$ if $|f| = o(|g|)$ and $f \approx g$ if for some $K > 0$, $K^{-1}|f| \le g \le K|f|$.

For the general theory of linear differential operators we refer to the books by Glazman [17], Naimark [30], and Weidmann [39]. In a sense this article may be considered an extension of [3].

This article is organized as follows: Introduction, spectral theory and asymptotic integration, the resolvent, conditions for $\sigma_{\text{ess}}(T_{\max}) \ne \mathbb{C}$, eigenvalues of equal magnitude, and other higher order equations.

2. Spectral theory and asymptotic integration

2.1. **Spectral theory.** For a closed, densely defined operator S in a Hilbert space $\mathcal{H}$, the *regularity field* is defined by

$$\Pi(S) = \{z \in \mathbb{C} : \|(S-z)(x)\| \ge k_z\|x\|,\ x \in D(S), \text{ for some } k_z > 0\}.$$

The *resolvent set* $\rho(S)$ of S is the set of all z in $\Pi(S)$ such that the range of $S - z$ is $\mathcal{H}$. The *spectrum* $\sigma(S)$ of S is the complement of $\rho(S)$. The set $\sigma(S)$ is the union three sets: the eigenvalues of S, $\sigma_p(S)$, the *residual spectrum* $\sigma_r(S)$ which is the set of values of $z \notin \sigma_p(S)$ for which the range of $S - z$ is closed but different from $\mathcal{H}$ and finally, the *essential spectrum* of S, $\sigma_{\text{ess}}(S)$ which is the set of z such that the range of $S - z$ is not closed. Glazman [17, p. 9] proves that this is equivalent (when there are no eigenvalues of infinite geometric multiplicity) to there being a singular sequence for z, i.e., a bounded noncompact sequence $\{f_n\}$ such that $(S - z)(f_n) \to 0$ as $n \to \infty$ or equivalently there is a sequence $\{f_n\}$ with $\|f_n\| = 1$ such that $(S - z)(f_n) \to 0$ as $n \to \infty$ and $f_n \to 0$ weakly.

If S is a $\mathcal{C}$-selfadjoint operator, then $\sigma_r(S) = \emptyset$ since $z \in \sigma_r(S)$ implies $N(S^* - \bar{z}) \ne \emptyset$. (see (2.1) below) If $S^*\phi = \bar{z}\phi$, then $S\bar{\phi} = \mathcal{C}S^*\mathcal{C}\bar{\phi} = \mathcal{C}S^*\phi = \mathcal{C}(\bar{z}\phi) = z\bar{\phi}$

which is contrary to $z \in \sigma_r(S)$. Thus a $\mathcal{C}$-selfadjoint operator has no residual spectrum. This parallels the selfadjoint case as selfadjoint operators have no residual spectrum. One difference however, is that a $\mathcal{C}$-symmetric operator always has a $\mathcal{C}$-selfadjoint extension, see Knowles [24], while a symmetric operator may not have a selfadjoint extension. This occurs for symmetric operators with finite and unequal deficiency indices.

In general, $\sigma(S) = \sigma_p(S) \cup \sigma_r(S) \cup \sigma_{\mathrm{ess}}(S)$ and $\sigma(S) = \sigma_p(S) \cup \sigma_{\mathrm{ess}}(S)$ if S is a $\mathcal{C}$-selfadjoint operator. While for both selfadjoint and $\mathcal{C}$-selfadjoint operators the spectrum is the union of the point and essential spectrum, there is an important difference. If S is selfadjoint, then both $\sigma(S) \neq \emptyset$ (by the spectral theorem) and $\rho(S) \neq \emptyset$ (as $z \in \rho(S)$ if z $\neq 0$). If S is a $\mathcal{C}$-selfadjoint operator, then it is possible for $\sigma(S) = \emptyset$ (see Example 2.2 below) or for $\rho(S) = \emptyset$ (see Example 2.1 below).

From the fact that $D(T_{\max})$ is a finite dimensional extension of $D(T_{\min})$, it can be proved , see [22, p. 16], that when one of $T_{\min} - z, T_{\max} - z, T^+_{\min} - \bar{z}, T^+_{\max} - \bar{z}$ has a closed range, then all do. From this it follows that

$$\sigma_{\mathrm{ess}}(T_{\min}) = \sigma_{\mathrm{ess}}(T_{\max}).$$

For $S = T_{\min}$, $I = [a, \infty)$, as noted before, we have $\sigma(T_{\min}) = \sigma_r(T_{\min}) \cup \sigma_{\mathrm{ess}}(T_{\min})$. We have the well known relations, Kato [21, p. 267],

$$N(T^+_{\max} - \bar{z}) = (R(T_{\min} - z))^{\perp}, \quad N(T_{\max} - z) = (R(T^+_{\min} - \bar{z}))^{\perp}. \tag{2.1}$$

Note that the conjugation map $y \to \bar{y}$ shows that $\dim N(T^+_{\max} - \bar{z}) = \dim N(T_{\max} - z)$. In the general case studied here the numerical range of $T_{\min}$ may be $\mathbb{C}$.

For $z \notin \overline{N(T_{\min})}$, we have from Kato [21, p. 268] that $T_{\min} - z$ has a closed range, nullity $T_{\min} - z = 0$, and the defect of $T_{\min} - z$ is constant on each connected component of $\overline{N(T_{\min})}^C$. Thus one has $\sigma_{\mathrm{ess}}(T_{\min}) \subseteq \overline{N(T_{\min})}$.

If $z \notin \sigma_{\mathrm{ess}}(S)$ and $z \notin \sigma_p(S)$, then by the closed graph theorem, $z \in \Pi(S)$; the converse also holds so that

$$\begin{gathered}\Pi(S) = \sigma_{\mathrm{ess}}(S)^C \cap \sigma_p(S)^C \text{ and} \\ \mathbb{C} = \Pi(S) \cup \Pi(S)^C = \Pi(S) \cup \sigma_{\mathrm{ess}}(S) \cup \sigma_p(S).\end{gathered} \tag{2.2}$$

We define

$$s = \dim(D(T_{\max})/D(T_{\min})). \tag{2.3}$$

In the one singular endpoint case $I = [a, \infty), s \geq 2n$ since one can construct $2n$ compactly supported independent functions in $D(T_{\max})/D(T_{\min})$ [30]. Further, it follows in the one singular endpoint case from Kauffman, Read, and Zettl [22, p. 16], that when $T_{\min} - z$ has a closed range,

$$s = \mathrm{nul}(T_{\max} - z) + \mathrm{nul}(T^+_{\max} - \bar{z}) = 2\,\mathrm{nul}(T_{\max} - z). \tag{2.4}$$

From these in this case and for all $z \notin \sigma_{\mathrm{ess}}(T_{\min})$, we obtain

$$\mathrm{def}(T_{\min} - z) := \dim(R(T_{\min} - z))^{\perp} \geq n, \quad \text{and } \mathrm{def}(T^+_{\min} - \bar{z}) \geq n. \tag{2.5}$$

For a $\mathcal{C}$-symmetric operator these defect numbers are independent of $z \notin \sigma_{\mathrm{ess}}(T_{\min})$ [25], and we refer to them as def $T_{\min}$ and def $T^+_{\min}$.

Example 2.1. McLeod [28] gave the example of the equation

$$\tau[y] := -y'' - 2i(\exp(2(1+i)x))y = zy, \quad 0 \leq x < \infty,$$

whose solutions can be expressed in terms of Bessel functions, and no nontrivial solution is in $\mathcal{L}^2([0, \infty))$ for any z. If $z \notin \sigma_{\mathrm{ess}}(T_{\min}) = \sigma_{\mathrm{ess}}(T_{\max})$ for some $z \in \mathbb{C}$, then (2.4)-(2.5) yields that $\operatorname{nul}(T_{\max} - z) \geq 1$ which is a contradiction. Thus $\sigma_{\rm ess}(T_{\min}) = \mathbb{C}$ which also implies $\overline{N(T_{\min})} = \mathbb{C}$ as $\sigma_{\rm ess}(T_{\min}) \subseteq \overline{N(T_{\min})}$.

This result holds for powers of τ as well. For simplicity, consider τ^2. First we show τ^2 has no eigenvalues. Suppose

$$\tau^2[y] = z^2 y, \quad y \neq 0, \quad y \in \mathcal{L}^2([0,\infty)).$$

Then

$$\tau^2[y] - z^2 y = (\tau - z)(\tau + z)[y] = 0.$$

This implies $g := (\tau + z)[y] \neq 0$ as $\tau + z$ has no nontrivial solutions in $\mathcal{L}^2([0,\infty))$ which in turn implies $(\tau - z)[g] \neq 0$ as $\tau - z$ has no nontrivial solutions in $\mathcal{L}^2([0,\infty))$ which is a contradiction. Since $\dim D(\hat{T}_{\max}/D(\hat{T}_{\min}) \geq 4$, where $\hat{T}$ refers to τ^2, we reach a contradiction as before Hence there are examples of even powers of differential operators with essential spectrum $\mathbb{C}$.

Example 2.2. In [8], the eigenvalue problem on $0 \leq x \leq 1$,

$$\tau[y] = -y'',$$

with boundary conditions

$$A \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} + B \begin{bmatrix} y(1) \\ y'(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \operatorname{rank}[A, B] = 2,$$

with complex matrices A, B was proved to be $\mathcal{C}$- symmetric if an only if the boundary conditions are of the form

$$y(0) = -cy(1), \quad y'(0) = cy'(1), \quad c = \pm i.$$

Further, it was shown than with $c = \pm i$ there are no eigenvalues. Hence we have an example of a $\mathcal{C}$-symmetric operator with empty spectrum as the minimal operator for a compact interval has empty essential spectrum.

Example 2.3. First we recall how singular sequences are used to find points in the essential spectrum. To show the minimal operator $T_{\min}$ for $\tau[y] = (-1)^n y^{(2n)}$, $0 \leq x < \infty$, has $\sigma_{\rm ess}(T_{\min}) = [0, \infty)$, one can construct a singular sequence. For example, let $\{I_n\}$, $I_n = [a_n, b_n]$, be a sequence of disjoint intervals in $[0, \infty)$ so that $3 \leq |I_n| = b_n - a_n$ and $|I_n| \to \infty$ as $n \to \infty$. Let ϕ_n be a C_∞ function with support I_n so that $\phi_n(x) = 1$ on $[a_n + 1, b_n - 1]$, $n = 1, 2, \dots$.

Let $y_n(x)) = \phi_n(x) \sin(\lambda^{1/2n} x)$ for $\lambda > 0$. Then for $\phi_n(x) \equiv 1$,

$$(T_{\min} - \lambda)[y_n] = (-1)^n y_n^{(2n)} - \lambda y_n = \lambda y_n - \lambda y_n = 0.$$

and it is clear that

$$||(T_{\min} - \lambda)[y_n]||/||y_n|| \to 0 \quad \text{as } x \to \infty,$$

so that $\{y_n\}$ is a singular sequence for $T_{\min}$ establishing $(0, \infty) \subseteq \sigma_{\rm ess}(T_{\min})$. Since $\sigma_{\rm ess}(T_{\min})$ is closed and $\mathcal{N}(T_{\min}) \subseteq [0, \infty)$, this gives $\sigma_{\rm ess}(T_{\min}) = [0, \infty)$. If $c \in \mathbb{C}$, and $f(x) = c$ on the supports of the ϕ_n and zero elsewhere, then the same singular sequence as above shows that $\sigma_{\rm ess}((T + f)_{\min}) = \mathcal{R}(c)$ where $\mathcal{R}(c)$ is the ray

$$\mathcal{R}(c) = \{\mu : \mu = \lambda + c, \, \lambda \geq 0\}.$$

That $\sigma_{\rm ess}((T + f)_{\min}) = \mathcal{R}(c)$ follows from (4.3) below, but the construction above shows how singular sequences generate points in the essential spectrum.

Now suppose K is a convex set in $\mathbb{C}$ and let $\{c_n\}$ be a countable sequence whose closure is K. Suppose $[0, \infty)$ is decomposed into disjoint intervals I_{ij} such that

$|I_{ij}| \to \infty$ as $i \to \infty$ and $|I_{ij}| \to \infty$ for $j \to \infty$. Define the potential V by $V(x) = c_i$ if $x \in I_{ij}$.

By the above argument we see that for all n, $\mathcal{R}(c_n) \subseteq \sigma_{\rm ess}((T+V)_{\min})$. Since the essential spectrum is closed,

$$\overline{\cup_{(n)}\mathcal{R}(c_n)} \subseteq \sigma_{\rm ess}((T+V)_{\min}).$$

Note the left side of this equation is a closed convex set which is also contained in the closure of the numerical range of $(T+V)_{\min}$ which is itself a convex set. This example illustrates the variety of sets that can be essential spectrum for Sturm-Liouville operators with complex coefficients. In particular, if K is the left half plane, then $\sigma_{\rm ess}((T+V)_{\min}) = \mathbb{C}$.

2.2. **Asymptotic integration.** Asymptotic integration has been a major tool to derive properties of the eigenfunctions of the maximal $\mathcal{C}-$ symmetric extensions of $T_{\min}$. In the beginning it was mainly used to compute the deficiency index of differential operators. The application to spectral theory began with [3]. Edmunds and Evans [13] gave five Fredholm type definitions of essential spectrum. For $\mathcal{C}-$ symmetric operators, the first four define the same object. The eigenvalues define the point spectrum $\sigma_p(T)$. It is obvious that $\sigma_{\rm ess}(T)$ will only depend on the asymptotics of the coefficients of T. In particular it will generally be independent of the boundary conditions at the left endpoint.

In asymptotic integration one first writes

$$Ty = zy$$

in systems form. With the quasi-derivatives (1.2) and

$$u = \left[y^{[0]}, y^{[1]}, \dots, y^{[n-1]}, y^{[2n-1]}, \dots, y^{[n]}\right]^t, \tag{2.6}$$

where t is transpose, the equation $Ty = zy$ can be written as as a system

$$\mathcal{J}u' = [z\mathfrak{A} + \mathfrak{B}]u \tag{2.7}$$

where

$$\mathcal{J} = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \quad \mathfrak{A} = \operatorname{diag}[w, 0, \dots, 0], \quad \mathfrak{B} = \begin{bmatrix} -C & A^t \\ A & B \end{bmatrix},$$

and the nonzero elements of A, B, C are [2] (Note that $B = B^t$, $C = C^t$)

$$A_{i,i+1} = 1, \quad B_{nn} = 1/p_n, \quad C_{ii} = p_{i-1}.$$

We will write the system (2.7) for short as

$$u' = \mathfrak{C}u \quad \text{with } \mathfrak{C} = \begin{bmatrix} A & B \\ \tilde{C} & -A^* \end{bmatrix} \tag{2.8}$$

where $\tilde{C}$ is C modified by replacing $C_{11} = p_0$ with $p_0 - zw$. This system formulation is similar, but different from that of [12, p. 105]. In fact, if G is a constant nonsingular matrix, then multiplying (2.8) by G gives an equivalent system with $\mathfrak{C}$ replaced by $G\mathfrak{C}G^{-1}$.

For asymptotic integration of (2.7) or (2.8) this system has to be brought into Levinson form [12],

$$v' = [\Lambda + R]v \quad \text{with} \Lambda = \operatorname{diag}[\lambda_i(x.z)], \quad R_{ij} = \mathfrak{L}^1, \quad i, j = 1, \dots, 2n, \tag{2.9}$$

because solutions of (2.9) will almost look like the solutions of the unperturbed system $v' = \Lambda v$ if Λ satisfies the dichotomy conditions. To transform (2.8) into (2.9) the matrix $\mathfrak{C}$ has to be diagonalized. Assume

$$S^{-1}\mathfrak{C}S = \Lambda.$$

Then the system for $Sv = u$ becomes

$$v' = \big(S^{-1}\mathfrak{C}S - S^{-1}S'\big)v = \big(\Lambda - S^{-1}S'\big)v. \tag{2.10}$$

Thus such a transformation makes sense if S, respectively $\mathfrak{C}$, is differentiable so that $S^{-1}S'$,i.e., $\mathfrak{C}'$ is small. This seems to restrict systems (2.8) to those with differentiable coefficients. However, if $\mathfrak{C}$ can be written as $\mathfrak{C} = \mathfrak{C}_1 + \mathfrak{C}_2$ with $\mathfrak{C}_1$ differentiable and $\mathfrak{C}_2$ integrable it suffices to diagonalize the smooth part $\mathfrak{C}_1$ of $\mathfrak{C}$. Such diagonalizations may be repeated. Thus, if one can write $\mathfrak{C} = \mathfrak{C}_0 + \mathfrak{C}_1 + \mathfrak{C}_2 + \dots + \mathfrak{C}_m$, with $\mathfrak{C}_k m - k$ times differentiable, $\mathfrak{C}_0$ a constant, $\mathfrak{C}_k^{(l)} = o(1)$, and $\mathfrak{C}_n^{(n)} \in \mathfrak{L}_0$, this will essentially lead to Levinson form. In addition note that it suffices to apply the diagonalization only to the off-diagonal parts of the systems matrix arising. For simplicity we shall restrict the analysis to $m = 3$ and require for the coefficients of $\mathfrak{C}$ a decomposition of the form

$$\begin{gathered} f = f_1 + f_2 + f_3 \text{ with } f_1 \text{ twice differentiable, } f_2 \text{ once differentiable and} \\ f_1''/(1+|f_1|), f_2'/(1+|f_1|), f + f_1'^2/(1+|f_1|), f_3 \in \mathfrak{L}_0^1. \end{gathered} \tag{2.11}$$

In this case we write

$$\mathfrak{C} = \mathfrak{C}_1 + \mathfrak{C}_2 + \mathfrak{C}_3 \tag{2.12}$$

and we restrict the diagonalization to $\mathfrak{C}_1 + \mathfrak{C}_2$. This is done in three steps.
I: Determine the eigenvalues of $\mathfrak{C}_1 + \mathfrak{C}_2$. II: Determine the eigenvectors.
III: Compute the diagonlaization matrix S.

As regards (I) we also have to require that all eigenvalues of $\mathfrak{C}_1 + \mathfrak{C}_2$ are distinct. Otherwise we may get a Jordan normal form type expressions which we will not study here. The eigenvalues of $\mathfrak{C}_1 + \mathfrak{C}_2$ are the roots of the characteristic polynomial of $\mathfrak{C}_1 + \mathfrak{C}_2$

$$\begin{aligned} P(\lambda, x, z) &= \sum_{k=1}^{n} (-1)^k p_k(x)\lambda^{2k} \ + (p_0(x) - zw) \\ &= p_n \det\big(\mathfrak{C}_1(x) + \mathfrak{C}_2(x) - \lambda\big) \end{aligned} \tag{2.13}$$

Here p_k are just the smooth parts of the coefficients. It is advantageous to replace λ by by $i\lambda$, Then one obtains the characteristic Fourier polynomial

$$P_F(\lambda, x, z) = P(i\lambda, x, z) = \sum_{k=1}^{n} p_k(x)\lambda^{2k} \ + (p_0(x) - zw). \tag{2.14}$$

For the remainder we will work with P_F only. Note that P_F is a function of λ^2, so that $-\lambda$ is an eigenvalue if λ is. Note that the Fourier polynomial of (2.7) is given by

$$P_F(\lambda, x, z) = -p_n \det\big(z\mathfrak{A} + \mathfrak{B} - i\lambda\mathcal{J}\big). \tag{2.15}$$

II: If the eigenvalues $\lambda_k = \lambda_k(x,z)$, $k = 1,\dots,2n$, of (2.13) are distinct, the eigenvectors $\rho_k(x,z)$ of $\mathfrak{C}\rho = \lambda\rho$ are given by [3]

$$\begin{gathered}(\rho_k)_s(x,z) = \lambda_k(x,z)^{s-1}, \quad 1 \le s \le n, \\ (\rho_k)_{n+s}(x,z) = \sum_{\nu=s}^{n} (-1)^{s+\nu} p_\nu(x)\lambda_k(x,z)^{2\nu-s}, \quad 1 \le s \le n.\end{gathered} \tag{2.16}$$

III: It is advantageous to base the diagonalization on the eigenvetors ν with

$$\nu_k(x,z) = M_k^{-1/2}(x,z)\big[\rho_1,\rho_2,\dots,\rho_n,\rho_{2n},\dots,\rho_{n+1}\big]^t(x,z) \tag{2.17}$$

with $M_k = \frac{\partial p_F}{\partial\lambda}\big|_{\lambda=\lambda_k}$ because the formulas for the selfadjoint case extends directly to this situation. Thus

$$S(x,z) = \big[\nu_1(x,z),\nu_2(x,z),\dots,\nu_{2n}(x,z)\big] \tag{2.18}$$

will be used as the diagonalizing matrix.

In the situation envisaged above, S diagonalizes $\mathfrak{C}_1 + \mathfrak{C}_2$ and $Sv = u$ leads to

$$v' = \big[\Lambda - S^{-1}S' + S^{-1}\mathfrak{C}_3 S\big]v, \quad \Lambda = \operatorname{diag}\big(\lambda(x,z)\big). \tag{2.19}$$

The matrix $S^{-1}S'$ in (2.19) is given below; similar formulas may be found in Eastham [12],

$$\begin{gathered}(S^{-1}S')_{jk} = \big(\lambda_j - \lambda_k\big)^{-1} M_j^{-1/2} M_k^{-1/2} \sum_{l=0}^{n} p_l \lambda_k^l \lambda_j^l, \quad j \ne k, \\ (S^{-1}S')_{kk} = 0.\end{gathered} \tag{2.20}$$

This latter relation is a consequence of the normalization of eigenvectors with $M_k^{-1/2}$. Note that these formulas also extend to the case where T has odd components.

The transformed system (2.19) is

$$Sv = u, \quad v' = \big(\Lambda + Q + R\big)v \tag{2.21}$$

where Q is the smooth off diagonal part of $-S^{-1}S'$ and where R is the remainder. If the terms of Q are sufficiently small, $Q_{ij} = o(1)$, a further matrix of the form $(I+B)$ may be applied. For this one needs that the eigenvectors in Λ are sufficiently distinct, With the usual perturbation Ansatz,

$$\big(I + \epsilon B\big)^{-1}\big(\Lambda + \epsilon Q\big)\big(I + \epsilon B_! + \epsilon^2 B_2 + \dots\big) = \Lambda + \epsilon\Lambda_1 + \epsilon^2\Lambda_2 + \dots,$$

one gets in this case for B_1,

$$B_{ii} = 0, \quad B_{ij} = \big(\lambda_j - \lambda_k\big)R_{ij}, \quad \Lambda_1 = 0. \tag{2.22}$$

Higher order corrections are of the order $O(\lambda_j - \lambda_k\big)^{-2} Q_{ij}\dots Q_{lk}$. This requires, for example,

$$|\lambda_i - \lambda_j| \ge \epsilon > 0, \quad Q_{ij} \in \mathfrak{L}_0^1. \tag{2.23}$$

The correction terms are then $(I+B)^{-1}B'$. These are integrable if $B'_{ij} \in \mathfrak{L}^1$.

If the remainder terms B, B', BB' are integrable, the system is in Levinson form. If not, further transformations as above may be necessary. While the application of asymptotic integration to problems of the deficiency index are rather straightforward. The use for spectral theory requires that the transformations above can be performed uniformly in the spectral parameters at least for z in a small neighborhood of a given z_0.

2.2.1. *Transformations.* To reduce the multitude of cases somewhat and to simplify the rising expressions, one should transform the variables. The best transformation we know is the Kummer-Liouville transformation which is based on [1, 2].

$$y(x) = \mu(t)z(t), \quad dt/dx = \gamma(x). \tag{2.24}$$

For differential equations of order six or higher the transformed coefficients are practically impossible to compute. For this reason we had introduced a transformation adapted to asymptotic integration [2], i.e., modulo Levinson terms the transformed system has the same form as the original system. For this one requires

$$\begin{gathered}\mu = \mu_1 + \mu_2, \gamma = \gamma_1 + \gamma_2, \mu_1, \gamma_1 \text{ twice differentiable, } \mu_2, \gamma_2 \text{ once differentiable },\\ \mu_1'/\mu\gamma,\ \gamma_1'/\gamma = o(1), \quad \gamma''/\gamma^2, \mu_1''/\mu\gamma \in \mathcal{L}(I).\end{gathered} \tag{2.25}$$

Then we let

$$b_k = k\mu_1'\gamma^{k-1} + \frac{1}{2}k(k-1)\mu\gamma_1'\gamma^{k-2}. \tag{2.26}$$

For the transformed coefficients one gets

$$\tilde{p}_n = \mu^2\gamma^{2n-1}p_n, \quad \tilde{p}_k = \mu^2\gamma^{2k-1}p_k - \mu\gamma^{k-1}\big(b_{k+1}p_{k+1}\big)' + b_k b_{k+1}p_{k+1}, \tag{2.27}$$

for $k = 0, \dots, n-1$.

If $p_n > 0$, the expressions for μ and γ can be obtained from $\tilde{p}_n = 1$ and $\tilde{w} = \mu^2 w/\gamma$. In this case the transformation is even unitary. If p_n is not positive, base the transformation on a smooth approximation of $|p_n|$.

2.2.2. *Dichotomy condition.* Levinson's Theorem states that a system on $[a, \infty)$,

$$v' = \big(\Lambda + R\big)v, \quad \Lambda = \operatorname{diag}\big[\lambda_i\big], \quad R \in \mathfrak{L}^1([a,\infty), \tag{2.28}$$

has solutions which almost look like the solutions of the unperturbed system $u' = \Lambda u$ if R is a small and if the λ_k satisfy a dichotomy condition, i.e.,

$$v_k(x) = \big(e_k + r_k(x)\big)\exp\Big(\int^x \lambda_k(t)\,dt\Big), \quad r_k(x) = o(1) \text{ as } x \to \infty, \tag{2.29}$$

where e_k is the unit vector with 1 as the kth component. The dichotomy condition requires the exponential terms to grow at sufficiently different rates. Details may be found in the book of Eastham [12], which is entirely devoted to Levinson's theorem. The dichotomy condition requires that for any pair of distinct indices $i, j \in \{1, \dots, n\}$ and for all $a \le x \le t < \infty$, and for some constants K_1, K_2,

$$\begin{gathered}\exp\Big(\int_t^x \operatorname{Re}(\lambda_i(s) - \lambda_j(s))\,ds\Big) \le K_1 \text{ or}\\ \exp\Big(\int_t^x \operatorname{Re}(\lambda_i(s) - \lambda_j(s))\,ds\Big) \ge K_2.\end{gathered} \tag{2.30}$$

Note that we are using the Fourier polynomial so that λ will have to be replaced with $i\lambda$ so that $\operatorname{Re}(\lambda_i(s) - \lambda_j(s)) = \operatorname{Im}(i\lambda_i(s) - i\lambda_j(s))$.

If system (2.8) has been transformed into Leviinson's form and if the dichotomy holds, then the solutions of (2.7) are given by

$$u_k(x,z) = S(x,z)\big(I + B(x,z)\big)\left[e_k + r_k(x,\lambda_k,z)\right]\exp\Big(\int_a^x \lambda_k(t,z)\,dt\Big), \tag{2.31}$$

with $r_k = o(1)$.

If further diagonalizations have been used, $(I+B)$ will have to be replaced by their product. If $B=o(1)$ formula (2.31) can be refined to

$$\begin{gathered} u_k(x,z)=M_k^{-1/2}(x,z)S(x,z)\big(e_k+r_k(x,\lambda_k,z)\big)\exp\Big(\int_a^x \lambda_k(t,z)dt\Big), \\ M_k=\partial p_F/\partial\lambda\big|_{\lambda=\lambda_k}. \end{gathered} \tag{2.32}$$

If one follows the proof of the asymptotic integration [12], one finds that the corrections terms $(S^{-1}S')_{jk}$ in (2.20) are z- uniformly bounded and $(S^{-1}S')_{jk}\to 0$ as $x\to\infty$ and as $|z|\to\infty$. This also holds for the correction terms arising in further diagonalizations. Following the proof of Levinson's Theorem [12] then shows that this extends to the corrections term $r_k(x,\lambda_k,z)$ as well. Thus

$$\begin{gathered} r_k(x,\lambda_k,z)\to 0 \quad \text{as } x\to\infty, \text{ uniformly in } z \text{ and} \\ r_k(x,\lambda_k,z)\to 0 \quad \text{as } |z|\to\infty. \end{gathered} \tag{2.33}$$

These results clearly show the importance of the first diagonalization. Asymptotic integration will be successful if the eigenvalues dominate the remainder terms. In the following we will call the factors $M^{-1/2}$ form factors, a term borrowed from nuclear physics. Note that this whole procedure requires a combination of decay and smoothness for the coefficients as well as the dichotomy condition for the eigenvalues.

At this point one should realize that asymptotic integration is stable with respect perturbations of the coefficients p_n by terms $q_n\in\mathfrak{L}_0^1$. In fact these terms define a relatively compact perturbations so that the essential spectrum remains invariant.

3. Resolvent

The aim of this article is to study the spectrum of operators (1.1) via their resolvent. However, it is still an open question when a resolvent exists for general $\mathcal{C}$-symmetric differential operators. Examples for $\mathcal{C}$-symmetric Sturm-Liouville operators operators without a resolvent or with empty spectrum are known as noted earlier. So far the only general information is based on the numerical range. In the case of compact operators the existence of the resolvent can also be inferred from the structure of the domain. In the situation we are considering here, we will quite often construct the resolvent explicitly, see (3.1) below. The development below has been presently worked out for $\mathcal{C}$-symmetric Hamiltonian systems with almost constant coefficients. We will be mainly deal with operators on the half-line. Problems in $\mathbb{R}$ can be handled by the decomposition method.

For the remainder of this section we make the following hypothesis.

(H1) There is a z_0 in $\mathbb{C}$ so that $z_0\notin\sigma_{\rm ess}(T_{\min})$, and $\dim N(T_{\max}-z_0)=n$.

This implies that $T_{\min}-z_0$ is a Fredholm operator and hence $T_{\min}-z$ is a Fredholm operator in a neighborhood K_0 of z_0 [27]. Since the Fredholm index of $T_{\min}-z$ is constant in K_0 and $\dim N(T_{\max}-z)=\dim N(T^+_{\max}-\bar z)$, it follows that (H1) holds in K_0. Property (H1) holds for problems with almost constant coefficients [7], and for operators with compact resolvent.

Note that if the coefficients in (1.1) are real and (1.1) is in the limit point condition at infinity, then $\dim N(T_{\max}-z_0)=n.$ for any non real z_0.

The operator $T_{\min}$ associated to (1.1) is $\mathcal{C}$-symmetric. As in the symmetric situation, we are, however, looking at the spectral theory of maximal $\mathcal{C}-$symmetric extensions H. Such extensions can be obtained from $T_{\max}$ by imposing boundary conditions. In general these are given at a and possibly at infinity, but under(H1) we only impose them at a to define a $\mathcal{C}$-selfadjoint operator [25]. This does not work for a selfadjoint limit circle operator.

Since the operator is regular at a and its deficiency index is n, the boundary conditions at a for a $\mathcal{C}$-selfadjoint operator can be defined by the boundary matrix [4],

$$Y_\alpha(a,z) = \begin{pmatrix} \alpha_1 & \overline{-\alpha_2} \\ \alpha_2 & \overline{\alpha_1} \end{pmatrix} \tag{3.1}$$

with $\alpha_1^*\alpha_1 + \alpha_2^*\alpha_2 = I_n$ and $\alpha_1^t\alpha_2 = \alpha_2^t\alpha_1$. With this the boundary conditions for the point a become, with u as in (2.6),

$$(\alpha_1^*, \alpha_2^*)u(a) = 0, \tag{3.2}$$

and we define the operator T_α as T restricted to the domain

$$D(T_\alpha) = \{y \in D(T) | (\alpha_1^*, \alpha_2^*)u(a) = 0\}. \tag{3.3}$$

By Knowles [25], T_α is a $\mathcal{C}$-selfadjoint operator. If $z \in K_0$ and z is not an eigenvalue of T_α, then $z \in \rho(T_\alpha)$.

The fundamental matrix for (2.7) with initial conditions (3.1) will be denoted by Y_α. When we write the fundamental matrix Y_α of (1.1) satisfying (3.1) as

$$Y_\alpha(x,z) = (\Theta_\alpha(x,z), \Phi_\alpha(x,z)), \tag{3.4}$$

then Φ_α satisfies the boundary condition at a.

Similar to the fundamental matrix Y_α for T_α,we have a fundamental matrix for the adjoint system. Throughout the remainder the adjoint system will be marked with a tilde. Thus, see [8],

$$\tilde{Y}_\alpha = (\tilde{\Theta}_\alpha, \tilde{\Phi}_\alpha) \quad \text{and} \quad \tilde{Y}_\alpha(a,z) = \begin{pmatrix} \overline{\alpha_1} & -\alpha_2 \\ \overline{\alpha_2} & \alpha_1 \end{pmatrix} \tag{3.5}$$

will stand for the solution of the adjoint system. By checking initial conditions of the fundamental matrices it follows from the symmetry conditions $B = B^t$, $C = C^t$ of (2.7) (see [7]) that

$$\tilde{Y}_\alpha(x,z) = \overline{Y_\alpha(x,z)} \quad \text{and hence } \tilde{\Theta}_\alpha = \overline{\Theta_\alpha},\ \tilde{\Phi}_\alpha = \overline{\Phi_\alpha}. \tag{3.6}$$

The proof in [7, Proposition 3.1] gives the following for $z \in K$ when (H1) holds.

Lemma 3.1. (a) *Let $I = [a, \infty)$. For $z \in K_0$ and z not an eigenvalue of T_α there exists a unique n by n matrix $M_\alpha(z)$ so that $Y_\alpha(x,z)\binom{I_n}{M_\alpha(z)} = \Theta_\alpha(x,z) + \Phi_\alpha(x,z)M_\alpha(z) \in \mathcal{L}^2_{\mathfrak{A}}(I)$. Here Y_α is the fundamental matrix of (2.9) satisfying (2.18). M_α is analytic for $z \in \mathbb{C} \setminus \sigma(T_\alpha)$.*

(b) *The M-matrix of the adjoint problem $\tilde{M}$ satisfies $\tilde{M}_\alpha(z) = M_\alpha^*(z)$ and $M_\alpha(z) = M_\alpha^t(z)$.*

With M_α determined this way, most of the properties derived in [7, sect. 4] can be shown. To do so we fix again Y_α as the fundamental matrix of (2.9) with initial conditions (2.18) Similarly let $\tilde{Y}_\alpha$ be the adjoint system for T^+ with the adjoint initial conditions (3.5). Then

$$\tilde{Y}_\alpha^* \mathcal{J} Y_\alpha \equiv \mathcal{J}, \quad Y_\alpha^* \mathcal{J} \tilde{Y}_\alpha \equiv \mathcal{J}, \tag{3.7}$$

can be shown as in [7]. Write $Y_\alpha = (\Theta_\alpha, \Phi_\alpha)$ and $\tilde{Y}_\alpha = (\tilde{\Theta}_\alpha, \tilde{\Phi})_\alpha$ then

$$\chi = \Theta + \Phi M_\alpha \in \mathcal{L}^2_{\mathfrak{A}}, \quad \tilde{\chi} = \tilde{\Theta} + \tilde{\Phi} M_\alpha^* \in \mathcal{L}^2_{\mathfrak{A}}, \tag{3.8}$$

where we have deleted the initial value index α. From (3.7) one gets $\mathcal{J}Y_\alpha\mathcal{J}\tilde{Y}_\alpha^* = -I_n$ and as in [7, sect. 4] one can deduce for $z \in K_0$ and $z \in \rho(T_\alpha)$ that

$$G(z,x,t) = \begin{cases} \Phi(x,z)\tilde{\chi}^*(t,z), & a \le x \le t, \\ \chi(x,z)\tilde{\phi}^*(t,z), & a \le t < x, \end{cases} \tag{3.9}$$

and

$$\tilde{G}(z,x,t) = \begin{cases} \tilde{\Phi}(x,z)\chi^*(t,z), & a \le x \le t, \\ \chi(x,z)\Phi^*(t,z), & a \le t < x, \end{cases} \tag{3.10}$$

are the integral kernels or Green's functions of the resolvents $\mathcal{R}_z = (T_\alpha - z)^{-1}$, respectively $\tilde{\mathcal{R}_z} = (T_\alpha^+ - z)$, i.e.,

$$(\mathcal{R}_z f)(x) = \int_a^\infty G(z,x,t)\mathfrak{A}(t)F(t)dt.$$

where F is as in (4.5) below.

For Hamiltonian systems the integration is based on the weight matrix $\mathfrak{A}$. For scalar equations, which we are considering here, the matrix $\mathfrak{A}$ is diag$[w, 0, \dots 0]$, so that only the first component counts and the integration uses the weight function w.

4. Conditions for $\sigma_{\rm ess}(T_{\min}) \neq \mathbb{C}$

We saw in section 3 that in order to have a meaningful spectral theory, it was important for $\Pi(T_{\min}) \neq \emptyset$, or equivalently, (when $\sigma_p(T_{\min}) = \emptyset$), $\sigma_{\rm ess}(T_{\min}) \neq \mathbb{C}$. As we noted in section 2, this occurs if $\overline{\mathcal{N}(T_{\min})} \neq \mathbb{C}$. A simple criterion is that the values of the coefficients p_k are all bounded below or more generally lie in a convex cone.

Another case of $\sigma_{\rm ess}(T_{\min}) \neq \mathbb{C}$ which we will not use however is when nul$(T - z) = 2n$. Race [33] proved this case for the limit circle one singular endpoint case for the Sturm-Liouville equation to obtain $\Pi(T_{\min}) = \mathbb{C}$. This result has been extended by Niessen [31] to operators of order $2n$.

4.1. **Almost constant coefficient case.** We say a function f on $I = [a, \infty)$ is almost constant if it can be written as

$$f = f_0 + f_1 + f_2,\ f_0 = \text{constant},\ f_1 \to 0 \text{ as } x \to \infty,\ f_2 \in \mathcal{L}(I).$$

Such decompositions are considered only for functions that are locally integrable. These conditions which are weaker that those needed for asymptotic integration, but strong enough for an exponential dichotomy, see [20]. For (1.1) we then assume that $p_0, \dots, p_n, 1/p_n, w$ are almost constant coefficient with $w = w_0 + w_1$ only.

First we define the polynomial P_{F0} in (2.13) as in (2.14),

$$P_{F0}(\lambda, z) = \sum_{k=1}^n p_{k0}\lambda^{2k} + (p_{00} - zw_0). \tag{4.1}$$

Note that the Fourier polynomial of (2.7) is also given by

$$P_F(\lambda, x, z) = -p_{n0} \det\big(z\mathfrak{A}_0 + \mathfrak{B}_0 - i\lambda\mathcal{J}\big). \tag{4.2}$$

In this section we work with P_{F0} only and assume $p_{n0} = 1$ without lost of generality. In the constant coefficient case P_{F0} is of course independent of x, and for a given z the λ-roots of $P_{F0}(\lambda,z)=0$ give rise to solutions of the form $\exp(i\lambda x)\xi$ of $Ty=zy$. Also P_{F0} has the form

$$P_{F0}(\lambda,z)=\lambda^{2n}+a_{2n-1}(z)\lambda^{2n-1}+\dots+a_0(z),$$

where the coefficients $a_j(z)$ are polynomials of z. A root $\lambda(z)$ of $P_{F0}(\lambda,z)=0$ is a holomorphic function in any simply connected region where there are no multiple roots. Further the set of multiple roots of $P_{F0}(\lambda,z)=0$ is either a finite set or $\mathbb{C}$. For further discussion of $P_{F0}(\lambda,z)=0$ see Lemma 3.3 of [3] which is extracted from the discussion of algebraic functions in Knopp [23]. The set of multiple roots of P_{F0} is finite if the discriminant of P_{F0} is not identically zero which holds in particular if P_{F0} is irreducible. In the constant coefficient case it is sufficient to assume the discriminant is not identically zero.

Solutions with $\operatorname{Im}\lambda>0$ for $I=[a,\infty)$ may lead to bound states if the boundary conditions fit. On the other hand solutions with $\lambda\in\mathbb{R}$ lead to bounded generalized eigenstates. These may be converted into approximate eigenfunctions by smooth cutoffs. These functions are approximate eigenfunctions independent of the particular boundary conditions at 0. Thus in the case of constant coefficients it is proved in [7] that the essential spectrum $\sigma_{\rm ess}(T_{\max})=\Sigma$ is given by

$$\Sigma=\{z:P_{F0}(\lambda,z)=0 \text{ for some } \lambda\in\mathbb{R}\}. \tag{4.3}$$

Then an additional assumption is made:

The discriminant of P_{F0} is not identically zero .

Then it is shown in [7] that for almost constant coefficients,

$$\sigma_{\rm ess}(T_{\max})\subseteq\Sigma\cup\mathcal{E},$$

where $\mathcal{E}$ is the finite set of z where P_F has multiple roots. This is proved by showing that under these hypotheses there is an *exponential dichotomy* for (2.7), i.e., there is fundamental matrix W of (2.7) and a projection matrix Q of rank n, and positive constants $K_1,K_2,\alpha_1,\alpha_2$ such that for $t,s\in[a,\infty)$,

$$\begin{gathered}\|W(t)QW^{-1}(s)\|\le K_1\exp\big(-\alpha_1(t-s)\big)\quad\text{for } t\ge s,\\ \|W(t)(I-Q)W^{-1}(s)\|\le K_2\exp\big(-\alpha_2(s-t)\big)\quad\text{for } s\ge t.\end{gathered}\tag{4.4}$$

The first equation in (4.4) shows that the columns of $W(t)Q$ form an n dimensional subspace of $N(T_{\max}-z)$ so $\dim N(T_{\max}-z)\ge n$. To prove $\dim N(T_{\max}-z)=n$ we must prove that the first elements of the columns of $W(t)(I-Q)$ form a subspace that contains only the zero element of $N(T_{\max}-z)$. Let the $2n\times n$ matrix Γ be a basis for the subspace formed by the columns of $W(t)(I-Q)$. Suppose for some vector $c\ne0$ that the first element of $\Gamma c\in N(T_{\max}-z)$. Now $\Gamma=W(I-Q)C$ for some $2n\times n$ matrix C of rank n. By (4.4), for $s\ge t$,

$$\|W(t)(I-Q)W^{-1}(s)\Gamma(s)c\|\le K_2\exp\big(-\alpha_2(s-t)\big)\|\Gamma(s)c\|.$$

Since

$$W(t)(I-Q)W^{-1}(s)\Gamma(s)c=W(t)(I-Q)Cc=\Gamma(t)c,$$

we have

$$\|\Gamma(t)\|\le K_2\exp\big(-\alpha_2(s-t)\big)\|\Gamma(s)c\|\quad\text{for } s\ge t.$$

But $\Gamma c\in N(T_{\max}-z)$ implies there is a sequence $\{s_n\}$ with $s_n\to\infty$ and $\Gamma(s_n)c\to0$ as $n\to\infty$. This implies $\|\Gamma(t)c\|=0$ for all t and thus $c=0$ which is a contradiction. Summarizing, we have the following theorem.

Theorem 4.1. *If the coefficients of T are almost constant and P_{F0} is irreducible, then Hypothsis* (H1) *holds for all $z \notin \Sigma \cup \mathcal{E}$.*

4.2. **General case.** We now develop a criterion based on the asymptotic solutions of (1.1) which may apply when $\mathcal{N}(T_{\min}) = \mathbb{C}$. The nonhomegeous version of (1.1), $Ty = zy + f$ has the form

$$\mathcal{J}u' = [\mathfrak{A}z + \mathfrak{B}]u + \mathfrak{A}F, \tag{4.5}$$

where $F = \big[f, 0, \dots, 0\big]^t$. We now order the eigenvalues of the characteristic polynomial (2.13) as

$$\operatorname{Im} i\lambda_1 \le \operatorname{Im} i\lambda_2 \le \dots \le \operatorname{Im} i\lambda_n < 0, \quad \lambda_{n+k} = -\lambda_{n+1-k},\ k = 1, \dots, n. \tag{4.6}$$

Assuming the conditions for asymptotic integration, there are solutions u_k of (2.7),

$$\begin{gathered} u_k(x,z) = M_k^{-1/2}(x,z) S(x,z)\big(e_k + r_k(x, \lambda_k, z)\big) \exp\Big(\int_a^x \lambda_k(t,z)dt\Big), \\ M_k = \partial p_F/\partial\lambda\big|_{\lambda=\lambda_K}. \end{gathered} \tag{4.7}$$

The component $(u_k)_1$ is a solution of $Ty = zy$. We now prove that $(u_k)_1 \in \mathcal{L}_w^2[a, \infty)$ under the addition of some further hypotheses. We will see in the next section that these conditions hold for a large class of operators where the eigenvalues are of equal magnitude. The bounds assumed below will be found in terms of the coefficients of (1.1) in section 5.

For $k = 1, \dots, n$, assume that for some $\delta > 0$,

$$-\frac{\pi}{2} + \delta \le \phi_k(x,z) \le \frac{\pi}{2} - \delta, \tag{4.8}$$

where $\lambda_k(x,z) = -\gamma_k(x,z)e^{i\phi_k(x,z)}$, $\gamma_k(x,z) = |\lambda_k(x,z)|$ and

$$\frac{1}{|M_k(x,z)|^{1/2}} \le L_k(x,z)\gamma_k^{1/2}(x,z)[1 + o(1)] \tag{4.9}$$

with $w(x)L_k^2(x,z)$ bounded on $[a, \infty)$. Then from (4.7) and (4.9),

$$\begin{aligned} |(u_k)_1(x,z)| &= \Big|M_k^{-1/2}(x,z)\Big| \big(1 + o(1)\big) \exp\Big(-\int_a^x \gamma_k(t,z)\cos(\phi_k(z,t)dt)\Big) \\ &\le L_k(x,z)\big(1 + o(1)\big)\gamma_k^{1/2}(x,z) \exp\Big(-\int_a^x \gamma_k(t,z)\sin(\delta)dt\Big). \end{aligned} \tag{4.10}$$

where we have used $\lambda_k = -\gamma_k[\cos\phi_k + i\sin\phi_k]$ so that

$$\big|\exp\Big(\int_a^x -i\gamma_k(t,z)\sin\phi_k(t,z)\,dt\Big)\big| = 1. \tag{4.11}$$

Note that $\cos\phi_k(t,z) \ge \cos(\pi/2 - \delta) = \sin\delta$. Thus for some constant C, independent of x,

$$\begin{aligned} &\int_a^\infty w(x)|(u_k)_1(x,z)|^2\,dx \\ &\le C\int_a^\infty \gamma_k(x,z)\exp\big(-2\int_a^x \gamma_k(t,z)\sin(\delta)dt\big)dx \\ &= \frac{-C}{2\sin\delta}\exp\Big(-2\int_a^x \gamma_k(t,z)\sin(\delta)dt\Big)\big|_a^\infty \\ &= \frac{C}{2\sin\delta} < \infty; \end{aligned} \tag{4.12}$$

hence $(u_k)_1 \in \mathcal{L}^2_w[a,\infty)$ for $k = 1,\dots,n$.

We define now the fundamental matrix for (2.7),

$$U(z,x) = \big[u_1(x,z),\dots,u_{2n}(x,z)\big] \tag{4.13}$$

Consider the variation of constants formula for (4.5),

$$\begin{aligned} u(x) = &\int_a^x U(x,z)P_1U(t,z)^{-1}\mathcal{J}^{-1}\mathfrak{A}(t)F(t)\,dt \\ &- \int_x^\infty U(x,z)P_2U(t,z)^{-1}\mathcal{J}^{-1}\mathfrak{A}(t)F(t)\,dt, \end{aligned} \tag{4.14}$$

where

$$P_1 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

Then u is a pointwise solution of (4.5), and the first component y of u belongs to $\mathcal{L}^2_{\mathfrak{A}}[a,\infty)$ if $f \in \mathcal{L}^2_w[a,\infty)$ at least for f of compact (4.14) support. By considering only y, we see that y is of the form

$$y(x) = \int_a^\infty K(x,t)w(t)f(t)\,dt =: \mathcal{T}(f), \tag{4.15}$$

where K is the kernel function defined implicitly by (4.14).

We will prove below that the operator $\mathcal{T}$ defined by (4.15) is a bounded operator from $\mathcal{L}^2_w[a,\infty)$ to $\mathcal{L}^2_w[a,\infty)$. This implies $y \in \mathcal{L}^2_w[a,\infty)$ if $f \in \mathcal{L}^2_w[a,\infty)$. Since by the variation of constants formula, $(T_{\max} - z)y = f$, this proves $\mathcal{T}$ is one-to-one, $D(\mathcal{T}^{-1}) = R(\mathcal{T}) \subseteq D(T_{\max})$, and $T_{\max} - z$ is onto $\mathcal{L}^2_w[a,\infty)$. Thus $T_{\max} - z$ has a closed range and therefore $z \notin \sigma_{\rm ess}(T_{\max})$. It also proves that $\mathcal{T}^{-1}$ is a restriction of $T_{\max} - z$. To prove $\mathcal{T}$ is bounded we use a theorem of Okikiolu [32, p.190].

Theorem 4.2. *Let the measures on $X, Y \subseteq [a,\infty)$ be defined by $m_X(x) = w(x)dx$, $m_Y(y) = w(y)dy$, and let $K_0(x,y)$ be a measurable function on $[a,\infty)\times[a,\infty)$ such that*

$$\int_X |K_0(x,y)|dm_X(x) \le M_1^2, \quad a.e.,\ y; \quad \int_Y |K_0(x,y)|dm_Y(y) \le M_2^2, \quad a.e.,\ x$$

for some constants M_1, M_2. Let $\mathcal{T}_0$ be the integral operator defined on $\mathcal{L}^2_w(X)$ by

$$\mathcal{T}_0(f)(y) = \int_X K_0(x,y)f(x)\,dm_X(x). \tag{4.16}$$

Then $\mathcal{T}_0$ is a bounded operator from $\mathcal{L}^2_w(X)$ to $\mathcal{L}^2_w(Y)$ with $\|\mathcal{T}_0\| \le M_1M_2$.

To apply Okikiiolu's Theorem, we must first compute K in (4.15). We first see that for $t \le x$, $K(x,t)$ is the $(1,n+1)$ entry of $U(x,z)P_1U(t,z)^{-1}$, and for $x \le t$, $K(x,t)$ is the $(1,n+1)$ entry of $U(x,z)P_2U(t,z)^{-1}$. By (2.32),

$$\begin{aligned} U(x,z) = &S(x,z)\mathcal{M}(x,z)\big(I + o(1)\big) \\ &\times \operatorname{diag}\Big[\exp\Big(\int_a^x \lambda_1(t,z)dt,\dots,\exp\Big(\int_a^x \lambda_{2n}(t,z)\,dt\Big)\Big)\Big], \end{aligned} \tag{4.17}$$

where

$$\mathcal{M} = \operatorname{diag}\big[M_1^{-1/2},\dots,M_{2n}^{-1/2}\big].$$

The first row of S is all one's, and the last column of S^{-1} is given by [12, p. 106], $\big[1/M_1, \dots, 1/M_{2n}\big]^t$. From (4.17) we have

$$\begin{aligned} U(t,z)^{-1} = \operatorname{diag}\Big[&\exp\big(-\int_a^t \lambda_1(t,z)dt, \dots, \exp\Big(-\int_a^x \lambda_{2n}(t,z)\,dt)\Big)\Big] \\ &\times \big(1+o(1)\big)\mathcal{M}(x,z)^{-1}S(x,z)^{-1}. \end{aligned} \tag{4.18}$$

A calculation shows that for $t \le x$, the $(1, n+1)$ entry of $U(x,z)P_1U(t,z)^{-1}$ is

$$K(x,t) = K_{1,+}(x,t) + \dots + K_{n,+}(x,t), \tag{4.19}$$

where

$$K_{k,+}(x,t) = \frac{[1+o(1)]\exp\big(\int_t^x \lambda_k(s,z)\,ds}{\big[M_k(x,z)M_k(t,z)\big]^{1/2}} \tag{4.20}$$

A similar calculation shows that for $x \le t$, the $(1, n+1)$ entry of $U(x,z)P_2U(t,z)^{-1}$. is

$$K(x,t) = K_{1,-}(x,t) + \dots + K_{n,-}(x,t), \tag{4.21}$$

where, as $x \to \infty$,

$$K_{k,-}(x,t) = \frac{[1+o(1)]\exp\big(\int_t^x \lambda_{n+k}(s,z)\,ds}{\big[M_k(x,z)M_k(t,z)\big]^{1/2}}. \tag{4.22}$$

Now make the assumption, for $t \ge a$, $x \ge a$, where $M_k = \partial_\lambda P_F(\lambda_k)$,

$$\frac{1}{|M_k(x,z)M_k(t,z)|^{1/2}} \le \frac{N_k(x,z)\gamma_k(t,z)}{w(t)}\exp\Big(\int_t^x |o(\gamma_k(s,z))|ds\Big) \tag{4.23}$$

with $N_k(x,z)$ bounded on $[a,\infty)$. Hence, we have for some constant C, independent of x and t, such that

$$\begin{aligned} &\int_x^\infty \big|K_{k,+}(x,t)\big|w(t)\,dt \\ &\le C\int_x^\infty w(t)\frac{\big|\exp\big(\int_t^x \lambda_k(s,z)\big|\,ds}{\big|M_k(x,z)M_k(t,z)\big|^{1/2}}\,dt \\ &\le C\int_x^\infty N_k(x,z)\gamma_k(t,z)\exp\Big(-\int_x^t [1+o(1)]\gamma(s,z)\sin(\delta)ds\Big)\,dt \\ &\le \frac{-CN_k(x,z)}{\sin\delta}\exp\Big(-\int_a^x \gamma_k(t,z)\sin(\delta)dt\Big)\big|_a^\infty \\ &= \frac{CN_k(x,z)}{\sin\delta} < \infty. \end{aligned} \tag{4.24}$$

In a similar way, we find that

$$\int_a^x \big|K_{k,-}(x,t)\big|w(t)\,dt \le \frac{CN_k(x,z)}{\sin\delta} < \infty. \tag{4.25}$$

and thus

$$\int_a^\infty \big|K_k(x,t)\big|w(t)\,dt \le \frac{CN_k(x,z)}{\sin\delta} < \infty. \tag{4.26}$$

where

$$K(x,t) = K_1(x,t) + \dots + K_n(x,t).$$

Hence, since (4.26) holds for every k, and with $N = N_1 + \cdots + N_n$,

$$\int_a^\infty \big|K(x,t)\big|w(t)\,dt \le \frac{CN(x,z)}{\sin\delta} \le \sup_{x\ge a} \frac{CN(x,z)}{\sin\delta} < \infty. \tag{4.27}$$

This establishes the second integral of Theorem 4.2.

To establish the first integral of Theorem4.2, we make an assumption parallel to (4.23), i.e.,

$$\frac{1}{|M_k(x,z)M_k(t,z)|^{1/2}} \le \frac{\tilde N_k(t,z)\gamma_k(t,z)}{w(x)} \exp\Big(\int_t^x |o(\gamma_k(s,z)|ds\Big) \tag{4.28}$$

with $\tilde N_k(x,z)$ bounded on $[a,\infty)$. Repeating the above argument yields

$$\int_a^\infty \big|K(x,t)\big|w(x)\,dx \le \frac{C\tilde N(t,z)}{\sin\delta} \le \sup_{t\ge a} \frac{C\tilde N(t,z)}{\sin\delta} < \infty. \tag{4.29}$$

Hence Okikiolu's Theorem applies to the operator $\mathcal{T}$ defined in (4.15). This proves the following theorem.

Theorem 4.3. *Under assumptions* (4.8)*,* (4.9)*,* (4.23)*, and* (4.28)*, it follows that for such z the maximal operator $T_{\max}$ and the minimal operator $T_{\min}$ satisfy*

$$z \notin \sigma_{\rm ess}(T_{\max}) = \sigma_{\rm ess}(T_{\min}).$$

In addition to having $\mathcal{T}^{-1}$ a restriction of $T_{\max} - z$, we will now prove that $\mathcal{T}^{-1}$ is an extension of $T_{\min} - z$.

Theorem 4.4. *Under the assumptions of Theorem 4.3 and def $(T_{\min} - z) = n$, the operator $\mathcal{T}^{-1}$ is an extension of $T_{\min} - z$.*

Proof. Let $\tilde y \in D(T_{\min})$, $f = T_{\min}(\tilde y)$, and $y = \mathcal{T}(f)$. Then for $\hat y := \tilde y - y$, we have $(T_{\max} - z)(\hat y) = f - f = 0$. Now $\tilde y^{[i]}(a) = 0$, $i = 0,\dots,2n-1$, and if we prove $y^{[i]}(a) = 0$, $i = 0,\dots,2n-1$ then $\hat y \equiv 0$ by uniqueness of initial value problems and $\tilde y = y \in D(\mathcal{T}^{-1})$.

We use the form of the Lagrange identity used by Knowles [25, p. 207] for functions $y_1, y_2 \in D(T_{\max})$, i.e.,

$$[y_1,y_2] = \sum_{k=1}^n \big(y_1^{[k-1]}y_2^{[2n-k]} - y_1^{[2n-k]}y_2^{[k-1]}\big). \tag{4.30}$$

Let $f_1 = (T_{\max} - z)(y_1)$, $f_2 = (T_{\max} - z)(y_2)$, and let u, v be the vectors corresponding to y_1, y_2 as in (2.6). Then a computation shows that

$$u^tJv = -[y_1,y_2], \quad (u^tJv)' = y_1wT_{\max}(y_2) - y_2wT_{\max}(y_2). \tag{4.31}$$

With U as in (4.13) and using the fact that $U^tJU = C$ is a constant matric as shown in [7]. we write with $n\times n$ blocks,

$$U^tJU = C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}. \tag{4.32}$$

The elements of C_{11} are the Lagrange forms $[u_i,u_j]$, $i.j = 1,\dots,n$. Knowles [25, Lemma 4.8] gives that these forms are all zero under the condition $\mathrm{def}(T_{\min}-z) = n$. Hence $C_{11} = 0$. Now U is non singular since the first components of the vectors $u_1,\dots,u_{2n}$ form a basis for the solutions of $(T_{\max} - z)(y) = 0$. Hence C_{12}, C_{21} are non-singular and $U^tJU = C$ gives $U^{-1} = C^{-1}U^tJ$. Using $n \times n$ blocks for U, a computation yields

$$\begin{aligned} P_2U^{-1} &= P_2C^{-1}U^tJ \\ &= \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} C^{-1} \begin{bmatrix} U_{11}^t & U_{21}^t \\ U_{12}^t & U_{22}^t \end{bmatrix} J \\ &= \begin{bmatrix} 0 & 0 \\ C_{12}^{-1}U_{21}^t & -C_{12}^{-1}U_{11}^t \end{bmatrix}. \end{aligned} \tag{4.33}$$

We define the n vectors O_n, $\tilde{f}$, by

$$O_n = [0, \dots, 0]^t, \quad \tilde{f} = [wf, 0, \dots 0]^t.$$

Then from (4.14),

$$u(a) = -U(a,z) \int_a^\infty \begin{bmatrix} 0 & 0 \\ C_{12}^{-1}U_{21}^t & -C_{12}^{-1}U_{11}^t \end{bmatrix} \begin{bmatrix} O_n \\ \tilde{f} \end{bmatrix}. \tag{4.34}$$

Let $y_1, \dots, y_n$ be the first components of the vectors $u_1, \dots, u_n$. Then

$$\int_a^\infty U_{11}^t \tilde{f} = [< y_1, \bar{f} >, \dots, < y_n, \bar{f} >]^t. \tag{4.35}$$

Since $(T_{\max} - z)(y_i) = 0$ by taking conjugates we have $(T_{\max}^+ - \bar{z})(\bar{y_i}) = 0$. Hence by (2.1),

$$\bar{y}_i \in N(T_{\max}^+ - \bar{z}) = R(T_{\min} - z)^\perp$$

This gives

$$\langle y_i, \bar{f} \rangle = \overline{\langle \bar{y}_i, f \rangle} = 0,$$

and so $u(a) = 0$ and thus $y^{[i]}(a) = 0$, for $i = 0, \dots, 2n-1$. □

From (3.8) we have that $\dim N(T_{\max} - z) \geq n$. Since a nontrivial combination of functions not in $\mathcal{L}_w^2[a,\infty)$ may be in $\mathcal{L}_w^2[a,\infty)$, we have not proved that $\dim N(T_{\max} - z) = n$ even though we have n independent non $\mathcal{L}_w^2[a,\infty)$ functions. Under the assumptions (4.8), (4.9), and (4.23), it is not clear that $\dim N(T_{\max} - z) = n$.

For later purposes (Theorem 4.6 for the bound (4.45)) we may repeat the above proof, under the assumption

$$\frac{1}{|M_k(x,z)M_k(t,z)|^{1/2}} \leq \frac{V_{r,k}(x,z)\gamma(t,z)^{1/r}}{w(t)^{1/r}} \exp\Big(\int_t^x |o(\gamma_k(s,z)|ds\Big) \tag{4.36}$$

with $V_{r,k}(x,z)$ bounded on $[a,\infty)$, to show that for $r \geq 1$,

$$\begin{aligned} &\int_x^\infty \big|K_{k,+}(x,t)\big|^r w(t)\,dt \\ &\leq C \int_x^\infty V_{r,k}^r(x,z)\gamma(t,z) \exp\big(-r \int_x^t \gamma(s,z)\sin(\delta)ds\big)\,dt \\ &\leq \frac{-CV_{r,k}^r(x,z)}{r\sin\delta} \exp\Big(-r\int_a^\infty \gamma_k(t,z)\sin(\delta)dt\Big)\Big|_a^\infty \\ &= \frac{CV_{r,k}^r(x,z)}{r\sin\delta} < \infty, \end{aligned} \tag{4.37}$$

and similarly,

$$\int_a^x \big|K_{k,-}(x,t)\big|^r w(t)\,dt \le \frac{CV_{r,k}^r(x,z)}{r\sin\delta} < \infty, \tag{4.38}$$

and hence

$$\int_a^\infty \big|K_k(x,t)\big|^r w(t)\,dt \le \frac{CV_{r,k}^r(x,z)}{r\sin\delta} < \infty. \tag{4.39}$$

We can take $V_{r,k} = N_k\gamma^{1-1/r}/w^{1-1/r}$, but we use $V_{r,k}$ to have a simpler notation. It then follows that with $V_r = V_{r,1} + \dots + V_{r,n}$,

$$\int_a^\infty \big|K(x,t)\big|^r w(t)\,dt \le \frac{Cn^{r+1}V_r^r(x,z)}{r\sin\delta} < \infty. \tag{4.40}$$

Similar arguments also show that

$$\int_a^\infty \big|K(x,t)\big|^r w(t)^{r/2}\,dt \le \frac{Cn^{r+1}w(x)^{-1+r/2}V_r^r(x,z)}{r\sin\delta} < \infty. \tag{4.41}$$

Theorem 4.5. *Assume that* (4.8), (4.9), *and* (4.23) *hold, and that in* (4.23) $N(x,z_0) \to 0$ *as* $x \to \infty$, *then the operator* $\mathcal{T}$ *defined by* (4.15) *is a compact operator from* $\mathcal{L}_w^2([a,\infty))$ *to* $\mathcal{L}_w^2([a,\infty))$.

Proof. To show that $\mathcal{T}$ is compact, let the integral operators $\mathcal{T}_b$ and $\tilde{\mathcal{T}}_b$ be defined by the kernels $\chi_{[a,b]}(x)K(x,s)$ and $\chi_{[b,\infty)}(x)K(x,s)$, i.e.,

$$(\mathcal{T}_b f)(x) = \int_a^\infty \chi_{[a,b]}(x)K(x,s)w(s)f(s)\,ds, \tag{4.42}$$

and similarly for $\tilde{\mathcal{T}}_b$. Applying Theorem 4.2 to $\tilde{\mathcal{T}}_b$ and using (4.29) $Y = [b,\infty)$, and $X = [a,\infty)$, it follows that $\tilde{\mathcal{T}}_b$ has an arbitrary small norm if b is sufficiently large. Thus $\mathcal{T}$ is compact if $\mathcal{T}_b$ is compact as it is the limit in operator norm of compact operators. To see that $\mathcal{T}_b$ is compact we employ a similar decomposition writing $\mathcal{T}_b = \mathcal{T}_{b1} + \mathcal{T}_{b2}$ where

$$(\mathcal{T}_{b1} f)(x) = \int_a^{b'} \chi_{[a,b]}(x)K(x,s)w(s)f(s)\,ds,$$
$$(\mathcal{T}_{b2} f)(x) = \int_{b'}^\infty \chi_{[a,b]}(x)K(x,s)w(s)f(s)\,ds.$$

Now $\mathcal{T}_{b1}$ is compact since its kernel is continuous on the compact set $[a,b]\times[a,b']$. Repeating the argument above shows that $\mathcal{T}_{b2}$ has arbitrary small norm if b' is sufficiently large; thus $\mathcal{T}_b$ is compact. $\square$

Recall that a compact kernel operator is a Hilbert-Schmidt operator if the kernel K satisfies

$$\int_a^\infty \int_a^\infty |K(x,s)|^2 w(s)w)x)\,ds\,dx < \infty. \tag{4.43}$$

A compact operator $\mathcal{T}$ defined on a Hilbert space belongs to the Schatten class $\mathcal{C}_s$ for $1 \le s < \infty$ provided that $\sum_1^\infty \mu(\mathcal{T})^s < \infty$ where the $\mu(\mathcal{T})$ are the s-numbers of $\mathcal{T}$, i.e., eigenvalues of the compact operator $(\mathcal{T}\mathcal{T})^*)^{1/2}$. $\mathcal{C}_\infty$ is the class of compact operators, and $\mathcal{C}_1$ is the class of trace class operators. If $k < s$, then $\mathcal{C}_k \subset \mathcal{C}_s$, and the inclusion is proper. Thus for $s > 2$, a Schatten class $\mathcal{C}_s$ operator may fail to be Hilbert-Schmidt.

The theorem below gives an upper bound for the norm, $s \ge 2$, of the Schatten class operator $\mathcal{C}_s$ generated by (1.1).

Theorem 4.6. *Under the conditions (4.8), (4.9), (4.23), and $s \geq 2$, the operator $\mathcal{T}$ defined by (4.15) is a Schatten class $\mathcal{C}_s$ operator if (4.30) below holds for $s = 2$.*

$$\int_a^\infty V_s(x,z)^s w(x)\,dx < \infty, \tag{4.44}$$

If (4.30) holds for some $s > 2$, then there is a constant C, independent of s, such that the Schatten norm $\|\mathcal{T}\|_s$ of $\mathcal{T}$ satisfies

$$\|\mathcal{T}\|_s^s \leq C \int_a^\infty V_s(x,z)^s w(x)\,dx < \infty. \tag{4.45}$$

Proof. From (4.40) for $s = 2$, we have

$$\int_a^\infty \int_a^\infty \big|K(x,t)\big|^2 w(x)w(t)\,dt\,dx \leq \frac{Cn^{r+1}\int_a^\infty V_2^2(x,z)w(x)dx}{r\sin\delta} < \infty \tag{4.46}$$

which proves $\mathcal{T}$ is Hilbert-Schmidt if (4.30) holds.

To establish (4.45) for $s > 2$, we use a theorem of Russo [37]. In his theorem we use the fact that $K(x,t) = K(t,x)$. Define the kernel by

$$k(x,t) = w(x)^{1/2}K(x,t)w(t)^{1/2},$$

and the operator $\tilde{\mathcal{T}} : \mathcal{L}^2[a,\infty) \to \mathcal{L}^2[a,\infty)$ by

$$(\tilde{\mathcal{T}}g)(x) = \int_a^\infty k(x,t)g(t)dt.$$

Note that (4.46) implies that $k \in \mathcal{L}^2[a,\infty) \times \mathcal{L}^2[a,\infty)$ so that Russo's theorem applies.

Russo's Theorem states that the Schatten class $\mathcal{C}_s$ norm of $\tilde{\mathcal{T}}$ satisfies, using also $k(x,t) = k(t,x)$

$$\|\tilde{\mathcal{T}}\|_s \leq \|k\|_{\nu,s} = \Big(\int_a^\infty \Big(\int_a^\infty |k(x,t)|^\nu dt\Big)^{s/\nu} dx\Big)^{\nu/s}, \quad \frac{1}{s} + \frac{1}{\nu} = 1. \tag{4.47}$$

From (4.41) with $r = \nu$, we have for some constant C_1, independent of ν,

$$\begin{aligned}\Big(\int_a^\infty \big|K(x,t)\big|^\nu w(t)^{\nu/2}\,dt\Big)^{1/\nu} &\leq C_1\Big(w(x)^{-1+\nu/2}V_\nu^\nu(x,z)\Big)^{1/\nu}\\ &= C_1 w(x)^{(\nu-2)/2\nu}V_\nu^\nu(x,z)\end{aligned} \tag{4.48}$$

Thus

$$\begin{aligned}&\int_a^\infty \Big(\int_a^\infty |k(x,t)|^\nu dt\Big)^{s/\nu} dx\\ &= \int_a^\infty \Big(\int_a^\infty w(x)^{\nu/2}\big|K(x,t)\big|^\nu w(t)^{\nu/2}\,dt\Big)^{s/\nu} dx\\ &\leq C_1 \int_a^\infty w(x)V_\nu^\nu(x,z)dx\end{aligned} \tag{4.49}$$

which will yield (4.45) by (4.47) after we verify that $\|\mathcal{T}\|_t = \|\tilde{\mathcal{T}}\|_t$. To see this let $M : \mathcal{L}_w^2([a,\infty)) \to \mathcal{L}^2([a,\infty))$ be defined by $M(y) = w^{1/2}y$. Then $M^{-1}(g) = g/w^{1/2}$, and $\|M(y)\|_{\mathcal{L}^2([a,\infty))} = \|y\|_{\mathcal{L}_w^2([a,\infty))}$. Thus M is isomorphic from $\mathcal{L}_w^2([a,\infty))$ onto $\mathcal{L}^2([a,\infty))$. Since $\tilde{\mathcal{T}} = M\mathcal{T}M^{-1}$, it follows that $\tilde{\mathcal{T}}$ and $\mathcal{T}$ are unitarily equivalent and $\|\mathcal{T}\|_t = \|\tilde{\mathcal{T}}\|_t$. □

Theorem 4.7. *Assume the hypotheses of Theorems 4.4 and 4.5. Let T_α be as in* (3.3) *and assume z is not an eigenvalue of T_α and* (1.1) *has* def $T_{\min} = n$. *Then $T_\alpha - z$ has a compact resolvent. Further T_α has a Hilbert-Schimdt resolvent if* (4.30) *holds for $s = 2$.*

Proof. Note that (H1) holds for z as in Theorems 4.4 and 4.5, and by section 2, 0 belongs to the resolvents of $\mathcal{T}^{-1}$ and $T_\alpha - z$. By [21, Corollary 6.34 p. 188], $\mathcal{T} = (\mathcal{T}^{-1})^{-1}$ is compact if and only if $(T_\alpha - z)^{-1}$ is. Further. by [21, Lemma 6.38 p. 188], the difference $(T_\alpha - z)^{-1} - \mathcal{T}$ is a finite rank operator. Since a finite rank operator is both compact and Hilbert-Schmidt, the result follows. $\square$

5. Eigenvalues of equal magnitude

When the eigenvalues are of equal magnitude, the bounds (3.4), (4.23), and (4.36) can be found explicitly. To illustrate this, we apply a theorem of Eastham [12, p.108] which uses less general asymptotic hypotheses than those of Section 2, but for which the explicit expressions we need have already been made. Eastham's theorem allows us to estimate the factors M_k above. Note the ordering of the coefficients in [12] are reverse of ours, i.e., his $p_0, \dots, p_{n-1}, p_n$ is our $p_n, \dots, p_1, p_0 - zw$. Also the matrix S of [12] and S of section 2 both have the properties of the first row is all ones and the last column of S^{-1} have the same formula so the computation of the kernel K of (3.10) is the same.

We show how Eastham's Theorem can be applied to give the bounds (4.9), (4.23), (4.28), and (4.36). First we define

$$P = \Big(\frac{p_0 - zw}{p_n}\Big)^{1/2n}.$$

We formulate below Eastham's theorem in our notation.

Theorem 5.1 ([12, p.108])**.** *Let $w, p_r (0 \le r \le n)$ have locally absolutely continuous first derivatives in $[a, \infty)$, and let p_n and $p_0 - zw$ be nowhere zero in $[a, \infty)$. Also let for $r = 1, \dots, n-1$*

(i) $p_{n-r}/[p_n(x)P^{2r)}(x)] \to c_r$ *as $x \to \infty$, where c_r is a finite limit;*
(ii) *the polynomial ($c_0 = c_1 = 1$ in Eastham)*
$$g(\xi) = \xi^{2n} + c_1\xi^{2n-2} + \dots + c_{n-1}\xi^2 + 1 \tag{5.1}$$
have $2n$ distinct roots ξ_k $(1 \le k \le 2n)$;
(iii) $\frac{p'_{n-r}}{p_n} = o(P^{2r+1})$ *as $x \to \infty$, $r = 0, \dots, n-1$,* $\frac{p'_0 - zw'}{p_0 - zw} = o(P)$;
(iv) $\frac{p''_{n-r}}{p_n P^{2r+1}} \in \mathcal{L}[a, \infty)$, $r = 0, \dots, n-1$, $\frac{p''_0 - zw''}{(p_0 - zw)P} \in \mathcal{L}[a, \infty)$;
(v) $\frac{p'^2_r}{p_n^2 P^{4r+1}} \in \mathcal{L}[a, \infty)$, $r = 0, \dots, n$.

Finally, let $\mathrm{Re}[\lambda_j(x, z) - \lambda_k(x, z)$ *have only one sign in $[a, \infty)$ for each unequal pair j, k in $[1, 2n]$, where the λ_k are the solutions of*

$$p_n\lambda^{2n} + p_{n-1}c_1\lambda^{2n-1} + \dots + c_{n-1}p_1\lambda^2 + p_0 - zw = 0$$

Then $T[y] = \lambda y$ has solutions $y_k(x, z), (1 \le k \le 2n)$, such that, as $x \to \infty$,

$$y_k(x, z) = \big(p_n(p_0 - zw)^{2n-1}\big)^{-1/4n} [1 + o(1)] \exp\Big(\int_a^x \lambda_k(t, z) dt\Big). \tag{5.2}$$

Eastham's conditions arise from the leading terms in the Kummer-Liouville transformation. So the conditions in Theorem 5.1 just arise from operators with almost constant coefficients.

Lemma 5.2. *Let g, h be functions on $[a,\infty)$ such that $g(x) \neq 0$, $h(x) > 0, g$ is absolutely continuous, $g', h \in \mathcal{L}_{\rm loc}[a,\infty)$, and $|g'(x)/g(x)| = o(h(x)$ as $x \to \infty$. Then*

$$|\frac{g(x)}{g(t)}| \leq \exp\big|\int_t^x |o(h(s))|\,ds\big| \quad \textit{as } x,t \to \infty.$$

Proof. We have

$$|\frac{g(x)}{g(t)}| = \big|\exp\int_t^x \frac{g'(s)}{g(s)}\,ds\big| \leq \exp\big|\int_t^x |\frac{g'(s)}{g(s)}|\,ds\big| \leq \exp\big|\int_t^x |o(h(s))|\,ds\big|,$$

which completes the proof. □

First we note from [12] that

$$\lambda_k = P\xi_k[1+o(1)], \quad P = \big[p_0 - zw)/p_n\big]^{1/2n}, \tag{5.3}$$

and for some constant c_k,

$$M_k = c_k\lambda_k^{-1}p_nP^{2n}[1+o(1)] = c_k\lambda_k^{-1}(p_0 - zw)[1+o(1)]. \tag{5.4}$$

Hence

$$\frac{1}{|M_k(x,z)|^{1/2}} = \frac{|\lambda_k(x,z)|^{1/2}[1+o(1)]}{c_k|p_0(x) - zw(x)|^{1/2}} = \frac{\gamma_k(x,z)^{1/2}[1+o(1)]}{c_k|p_0(x) - zw(x)|^{1/2}}. \tag{5.5}$$

so we have

$$L_k(x,z) := \frac{1}{c_k|p_0(x) - zw(x)|^{1/2}},$$

Hence condition (4.9) becomes

$$\frac{w(x)}{c_k|p_0(x) - zw(x)|^{1/2}} \text{ is bounded on } [a,\infty). \tag{5.6}$$

Hence, $y_k \in \mathcal{L}_w^2[a,\infty)$ for $k = 1,\dots,n$, when (5.6) holds.

From (iii) above we have that $|p_n'/p_n| = o(\gamma)$ and $|(p_0' - zw')/(p_0 - zw)| = o(\gamma)$ so we can apply Lemma 5.2 with $g = p_n$, $h = \gamma$, and $g = p_0 - zw$, $h = \gamma$, respectively.

Further for some constant d_k,

$$\begin{aligned}
&\frac{1}{|M_k(x,z)M_k(t,z)|^{1/2}}\\
&= d_k\frac{[\gamma_k(x,z)\gamma_k(t,z)]^{1/2}[1+o(1)]}{|p_0(x) - zw(x)|^{1/2}|p_0(t) - zw(t)|^{1/2}}\\
&= d_k\frac{w(x)}{|p_0(x) - zw(x)|}\frac{w(t)}{w(x)}\big[\frac{\gamma_k(x,z)}{\gamma_k(t,z)}\big]^{1/2}\big|\frac{p_0(x) - zw(x)}{p_0(t) - zw(t)}\big|^{1/2}\frac{\gamma_k(t,z)}{w(t)}\\
&= d_k\frac{w(x)}{|p_0(x) - zw(x)|}\exp\big|\int_t^x |o(\gamma_k(s,z)|ds\big|\frac{\gamma_k(t,z)}{w(t)}
\end{aligned} \tag{5.7}$$

where we have applied Lemma 5.2. Thus one can define

$$N_k(x,z) := d_k\frac{w(x)}{|p_0(x) - zw(x)|}. \tag{5.8}$$

Hence condition (4.23) is

$$\frac{w(x)}{|p_0(x)-zw(x)|} \text{ is bounded on } [a,\infty). \tag{5.9}$$

Note that N_k and γ_k are independent of k except for a constant.

To compute V_r for $r\geq 1$, we will use $V_{r,k}=N_k\gamma_k^{1-1/r}/w^{1-1/r}$ mentioned earlier. Also because of the independence of k in N_k, γ_k, we use just V_r. Hence from (5.3) and (5.8),

$$\begin{aligned} V_r(x,z) &= (\text{const.})\frac{N_k(x)\gamma_k(t,z)^{1-1/r}}{w(x)^{1-1/r}} \\ &= (\text{const.})\frac{N_k(x)\gamma_k(x,z)^{1-1/r}}{w(x)^{1-1/r}}\,\frac{\gamma_k(t,z)^{1-1/r}}{\gamma_k(x,z)^{1-1/r}} \\ &= (\text{const.})\frac{w(x)^{1/r}}{|p_0(x)-zw(x)|}\Big|\frac{p_0(x)-zw(x)}{p_n(x)}\Big|^{\frac{r-1}{2rn}} \\ &= (\text{const.})\frac{w(x)^{1/r}}{|p_n(x)|^{\frac{r-1}{2rn}}|p_0(x)-zw(x)|^{\frac{2rn+1-r}{2rn}}}. \end{aligned} \tag{5.10}$$

where we have applied Lemma 5.2 as in (5.7). Thus we can define

$$V_r(x,z) := \frac{w(x)^{1/r}}{|p_n(x)|^{\frac{r-1}{2rn}}|p_0(x)-zw(x)|^{\frac{2rn+1-r}{2rn}}}. \tag{5.11}$$

Hence condition (4.36) is

$$\frac{w(x)^{1/r}}{|p_n(x)|^{\frac{r-1}{2rn}}|p_0(x)-zw(x)|^{\frac{2rn+1-r}{2rn}}} \text{ is bounded on } [a,\infty). \tag{5.12}$$

Thus (4.30) is

$$\int_a^\infty V_s(x,z)^s w(x)\,dx = \int_a^\infty \frac{w(x)^2}{|p_n(x)|^{\frac{s-1}{2n}}|p_0(x)-zw(x)|^{\frac{2sn+1-s}{2n}}}dx<\infty. \tag{5.13}$$

Finally, we come to the issue of (4.8). Here the roots of the polynomial (5.1) are needed. This is particularly simple for the two term equation, i.e., $p_r=0$, $r=1,\dots,n-1$. In this case, (5.1) is simply $\xi^{2n}+1=0$. For $n=1$: $\xi=\pm i$ For $n=2$: $\xi=\pm(1\pm i)/\sqrt{2}$, etc.

We examine the case $n=1$ more in detail when $w/p_0\to 0$ as $x\to\infty$. Suppose a root of $(p_0/p_1)^{1/2}$ satisfies

$$\delta\leq \arg(p_0/p_1\leq 2\pi-\delta \tag{5.14}$$

for some $\delta>0$. From (5.3),

$$\arg\lambda_k(x,z)=\Big(\arg\xi_k+\frac{1}{2}\arg\frac{p_0}{p_1}\Big)\Big(1+o(1)\Big) \tag{5.15}$$

Choosing $\xi_k=-i$, gives

$$\frac{-\pi}{2}+\frac{\delta}{2}\leq \arg\xi_k+\frac{1}{2}\arg\frac{p_0}{p_1}\leq\frac{\pi}{2}-\frac{\delta}{2} \tag{5.16}$$

Thus for some $\delta'>0$ and sufficiently large x,

$$\frac{-\pi}{2}+\delta'\leq\arg\lambda_k(x,z)\leq\frac{\pi}{2}-\delta'; \tag{5.17}$$

hence (4.8) holds. This agrees with the results of [8] when one takes into account that the leading coefficient there preceded by a minus sign.

Finally, we give a fourth order example to illustrate that Theorem 4.6 gives new results even for selfadjoint operators.

Example 5.3.

$$\tau[y] = y^{iv} + x^\alpha y, \quad 0 < a \le x < \infty. \tag{5.18}$$

This equation is known to be limit point at infinity, i.e., $\operatorname{def}(T_{\min} - z) = 2$, $\operatorname{Im} z \neq 0$, and to have spectrum that is discrete and bounded below. Let T_α be a selfadjoint operator generated by (5.18). As the coefficients are real, for all z non real, hypothesis (H1) holds. To apply Theorem 4.7, choose $n = s = 2$, $z = i$. The criteria for a Hilbert-Schmidt kernel for the resolvent $(T_\alpha - z)^{-1}$ with z non real by Theorem 4.7 is then

$$\int_a^\infty \frac{dx}{|x^\alpha - i|^{7/4}} < \infty \tag{5.19}$$

which is equivalent to $\alpha > 7/4$.

6. Other equations of higher order

The spectral analysis of higher order differential operators faces several difficulties. First of all the characteristic polynomial has to be factored. Then the dichotomy condition for the roots has to be shown. Finally, the eigenfunctions and resolvents have to be analyzed. The fourth order differential operators are somehow the gateway to higher order operators in as much new phenomena can first be observed for this class of operators. However, determining the roots of the characteristic polynomial is no problem at all, so that one can concentrate on the other critical phenomena. To avoid any technical difficulties, we consider only operators of the form (6.1) below. Here we only discuss the approach and refer to the literature for precise results in the case of real coefficients, and only indicate here how a similar approach may be carried out for operators of the form,

$$\tau[y] = (y'')'' + (p_1 y')' + p_0 y. \tag{6.1}$$

The characteristic polynomial

$$P_F(x, \lambda) = \lambda^4 + p_1 \lambda^2 + p_0 - z \tag{6.2}$$

has the roots, [12, p. 126],

$$\begin{gathered} \lambda_1 = -\lambda_2 = \frac{1}{\sqrt{2}}\sqrt{-p_1 + \Delta}, \quad \lambda_3 = -\lambda_4 = \frac{1}{\sqrt{2}}\sqrt{-p_1 - \Delta}, \\ \Delta = \sqrt{p_1^2 - 4(p_0 - z)} \end{gathered} \tag{6.3}$$

Of course we will also assume the usual properties of smoothness and decay (2.11) for the coefficients p_1 and p_0. Even though we have an explicit factorization of the Fourier polynomial, we will still have to demand the dichotomy condition, even though it is mostly easy to check in this case. The form factors

$$M_j = 4\lambda_j^3 + p_1 \lambda_j$$

are unbounded if p_0 is. In this case the operator has a compact resolvent. If p_0 and p_1 are bounded, then continuous spectrum may arise. The most interesting case is, when p_1 is dominant, i. e.,

$$(p_0 - z) = o(p_1^2). \tag{6.4}$$

In this case,

$$\Delta = p_1 - 2(p_0 - z)/p_1 + O((p_0 - z)^2/p_1^3)$$

so that

$$\begin{aligned} p_1^{-1/2}\lambda_1 &= \frac{1}{\sqrt{2}}\Big(-2\frac{p_0 - z}{p_1} + O\big(\frac{(p_o - z)^2}{p_1^2}\big)\Big)^{1/2}, \\ \lambda_2 &= \frac{1}{\sqrt{2}}\Big(-2p_1 + O\big(\frac{p_0 - z}{p_1}\big)\Big)^{1/2} \end{aligned} \tag{6.5}$$

In this case the dichotomy condition holds if it holds for λ_1, λ_2 and for λ_3, λ_4. The eigenvalues in the (1,2) block are proportional to $p_1^{-1/2}$, while the off block elements are proportional to $p_1' p_1^{-3/2}$. This means that a further diagonalization will turn these integrable expressions so that the problem is essentially that of blocks (1,2) and (3,4) Sturm-Liouville operators. One of these, the (2,4) block gives discrete spectrum.

This phenomenon can be observed for higher order operators with a dominant middle term. Even this can be generalized to operators with several classes of eigenvalues of different magnitude. For real coefficients this has bee carried out by Behncke and Nyamwala [5].

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Horst Behncke
Fachbereich Mathematik/Informatik, Universitat Osnabruck, 49069 Osnabruck, Germany
Email address: `sabine.schroeder@uni-osnabrueck.de`

Don Hinton
Mathematics Department, University of Tennessee, Knoxville, TN 37996, USA
Email address: `dhinton1@tennessee.edu`