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YAMABE BOUNDARY PROBLEM WITH SCALAR-FLAT MANIFOLDS TARGET

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Abstract. We present a survey on the compactness of the set of solutions for the Yamabe problem on manifolds with boundary. The stability of the problem is also discussed.

1. Introduction

In the early 1960's Yamabe [32] raised the famous question: given a compact Riemaniann manifold without boundary, is it possible to find a conformal metric which has constant scalar curvature? This problem, which carries the name of its author, had a wide echo in the mathematical community, and gave rise to a large number of related problems. For the history of the original Yamabe problem and its solution, we refer to the fundamental survey of Lee and Parker [26].

One possible generalization of the classical Yamabe problem is to consider manifolds with boundary. In this case, one asks if it is possible to find a conformal metric for which both the scalar curvature on the interior of the manifold and the mean curvature of the boundary are constant [23]. In this framework, a particular interesting case is when the target manifold is scalar flat. In fact, finding a scalar flat conformal metric with constant mean curvature of the boundary is not only a possible generalization of Yamabe problem, but can be also viewed as an extension of Riemann conformal mapping theorem to any dimension. This problem was firstly raised by Escobar in 1992 [11, 12], and was solved by Escobar himself, and by Marques, Chen, Brendle, and Almaraz [1, 6, 29, 31].

In this survey we study the compactness of the set of solutions for the Yamabe boundary problem when the target manifold is scalar flat. Also, we present some result on the effect of perturbation of the curvatures on the compactness of the the set of solutions.

2. Framework

We write the Yamabe boundary problem as an elliptic nonlinear partial differential equation. We recall that the conformal class of the metric g is

$$[g] = \{\bar{g} = u^{\frac{4}{n-2}} g, \text{ with } u \text{ positive and smooth on } M\},$$

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and that we can compute $R_{\bar g}$ and $h_{\bar g}$, respectively the scalar curvature on the manifold and the mean curvature of the boundary, respect to $\bar g = u^{\frac{4}{n-2}} g$, as

$$R_{\bar g} = \big[-\Delta_g u + \frac{n-2}{4(n-1)} R_g u \big] \frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}}$$
$$h_{\bar g} = \big[\frac{\partial}{\partial \nu} u + \frac{n-2}{2} h_g u \big] \frac{2}{n-2} u^{-\frac{n}{n-2}}$$

where R_g, h_g are respectively the scalar and the mean curvature referred to the original metric g, $-\Delta_g$ is the Laplace-Beltrami operator and ν is the outward unit normal vector to ∂M. With this in mind the problem is equivalent to finding a positive solution to

$$\begin{gathered} -\Delta_g u + \frac{n-2}{4(n-1)} R_g u = 0 \quad \text{in } M \\ \frac{\partial u}{\partial \nu} + \frac{n-2}{2} h_g u = (n-2) u^{\frac{n}{n-2}} \quad \text{on } \partial M \end{gathered} \tag{2.1}$$

or, in a more compact, and conformally invariant form,

$$\begin{gathered} L_g u = 0 \quad \text{in } M \\ B_g u + (n-2) u^{\frac{n}{n-2}} = 0 \quad \text{on } \partial M' \end{gathered} \tag{2.2}$$

where $L_g = \Delta_g - \frac{n-2}{4(n-1)} R_g$ and $B_g = -\frac{\partial}{\partial \nu} - \frac{n-2}{2} h_g$ are respectively the conformal Laplacian and the conformal boundary operator.

We remark that the nonlinearity $u^{\frac{n}{n-2}}$ in the boundary equation is critical for the immersion

$$H^1_g(M) \to L^t_g(\partial M).$$

Here, by $H^1_g(M)$ we refer to the Hilbert space which is the completion of the $C^\infty_0(M)$ function with respect to the norm $\|\cdot\|_H$ generated by the scalar product

$$\langle u, v \rangle_H = \int_M g(\nabla u, \nabla v) d\mu_g + \int_M uv d\mu_g,$$

and by $L^p_g(M)$ and $L^t_g(\partial M)$ we refer to usual Lebesgue spaces with respect to the volume form of the manifold (resp. the boundary of the manifold). In the following, where no ambiguity is possible, we refer to $g(\nabla u, \nabla v)$ simply as $\nabla u \cdot \nabla v$.

Problem (2.1) has a variational structure: a solution u of (2.1) is a critical point of the functional quotient

$$Q(M) := \frac{\int_M \big(|\nabla u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \big) dv_g + \int_{\partial M} \frac{n-2}{2} h_g u^2 d\sigma_g}{\big(\int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} d\sigma_g \big)^{\frac{n-2}{n-1}}}.$$

In analogy with the classical Yamabe problem, one can define the Yamabe quotient as

$$Q(M, \partial M) := \inf_{u \in H^1_g(M) \setminus \{0\}} Q(M)$$

and problem (2.1) has a unique solution if $-\infty < Q(M, \partial M) < 0$, a unique solution up to a constant factor if $Q(M, \partial M) = 0$, and at least one solution if $Q(M, \partial M) > 0$. For the existence part, we refer to the cited papers [1, 6, 11, 12, 29, 31].

The manifolds for which $Q(M, \partial M) > 0$ are called *manifolds of positive type*. Since the solution typically is not unique, in these manifolds it is interesting to study if the set of the solutions is compact for the C^2 convergence. That is, if there exists a priori bounds in $C^{2,\alpha}(M)$. The main result on compactness are cited in the next section.

3. Compactness

For the Yamabe problem on manifolds without boundary, the compactness of the set of solutions was proved, when M is not conformally equivalent to the round sphere, at first for low dimensions by Druet [8], Marques [30], Li and Zhang [27, 28]. Finally, in the celebrated paper by Khuri, Marques and Schoen [24], it was proved that compactness holds in dimension $3 \le n \le 24$ with the additional hypothesis of the positive mass theorem. Dimension $n = 25$ appears to be critical, in fact counterexamples to compactness exist for any $n \ge 25$ [5, 7].

The same dimension seems to play an important role also for the Yamabe boundary problem (2.2). In fact, Almaraz [3], adapted the paper of Brendle and Marques to prove that, for $n \ge 25$, there exists a smooth Riemannian metric g on the Euclidean ball B which is not conformally flat, for which ∂B is umbilic and there exists a sequence of positive smooth functions $\{u_n\}_n$ for (2.2) such that $\|u_n\|_{C^2(B)} \to +\infty$.

On the other hand, the question of compactness has not yet completely settled. The Khuri, Marques, and Schoen procedure has been adapted to the Yamabe boundary problem for scalar flat curvature in various settings but there are still open problems, and a general theorem for $n \le 25$ is missing. Throughout this section, we will examine the various results present in literature up to now. We will try to give an idea of the common strategy to achieve these results.

When dealing with compactness of the boundary Yamabe problem, we study the equation

$$\begin{gathered} L_g u = 0 \quad \text{in } M \\ B_g u + (n-2)u^p = 0 \quad \text{on } \partial M \end{gathered} \tag{3.1}$$

for $1 \le p \le \frac{n}{n-2}$. The reason to consider (3.1) is twofold. On the one hand, the technique to prove compactness for the critical nonlinearity extends, almost for free, to the subcritical case. On the other hand, this result of compactness provides also an alternative proof of the existence of solution for Yamabe problem.

In Yamabe problems, often low dimensions are special, and require different strategies. Indeed, for Yamabe boundary problems compactness holds without further assumptions in dimensions $n = 3, 4$. More precisely:

Theorem 3.1. *Let (M, g) be an n-dimensional manifold of positive type, not conformally equivalent to the standard ball. Let $n = 3, 4$. Then for each $\bar{p} \in (1, \frac{n}{n-2})$, there exists $C > 0$ such that if $p \in [\bar{p}, \frac{n}{n-2}]$ and u is a solution to* (3.1),

$$C^{-1} \le u \le C, \text{ and } \|u\|_{C^{2,\alpha}(M)} \le C.$$

for some $\alpha \in (0, 1)$.

This theorem has been proved for $n = 3$ by Almaraz, de Queiroz and Wang in [4], and for $n = 4$ by Kim, Musso, and Wei in [25].

For dimension $n \ge 5$ the geometry of the boundary appears to be important to rule out the possibility of blow up. Let us introduce the definition of umbilic and non umbilic boundary.

Definition 3.2. A manifold M has *non umbilic* boundary if the trace-free second fundamental form of ∂M is everywhere different from zero, while M has *umbilic* boundary if the trace-free second fundamental form of ∂M vanishes everywhere on the boundary.

When the boundary is non umbilic, it is possible to prove that, if a sequence u_n blows-up, that is if there exists a sequence $x_n \to x_0$ such that $u_n(x_n) \to +\infty$, then $x_0 \in \partial M$ and the trace-free second fundamental form vanishes at x_0. This is the key point of the following result, proved by Almaraz in [2] for dimension $n \geq 7$ and extended by Kim, Musso and Wei in [25] to dimensions $n = 5, 6$.

Theorem 3.3. *Let (M, g) be a Riemannian manifold of positive type with regular boundary and dimension $n \geq 5$. Assume that the trace-free 2nd fundamental form of ∂M is nonzero everywhere. Then for each $\bar{p} > 1$, there exists $C > 0$ such that for any $p \in [\bar{p}, \frac{n}{n-2}]$ and any $u > 0$, solution to equation (3.1),*

$$C^{-1} \leq u \leq C \text{ and } \|u\|_{C^{2,\alpha}(M)} \leq C.$$

The opposite situation is when the manifold has umbilic boundary. In this scenario, the first result is due to Felli and Ahmedou [13], which considers a locally conformally flat manifold, and uses the positive mass theorem, which in these manifolds is known to hold, to avoid the possibility of blow up. More precisely, their result is:

Theorem 3.4. *Let (M, g) be a locally conformally flat Riemannian manifold of positive type with regular umbilic boundary and dimension $n \geq 3$. Assume that (M, g) is not conformally equivalent to the standard ball. Then, given $\bar{p} > 1$, there exists $C > 0$ such that for each $p \in [\bar{p}, \frac{n}{n-2}]$ and $u > 0$, solution to the equation (3.1),*

$$C^{-1} \leq u \leq C \text{ and } \|u\|_{C^{2,\alpha}(M)} \leq C$$

for some $0 < \alpha < 1$.

When the boundary is umbilic, another approach is to assume that the Weyl tensor never vanishes on the boundary. This plays the same role of the tensor of the second fundamental form, and could be used to get the following result [14, 16].

Theorem 3.5. *Let (M, g) a smooth, n-dimensional Riemannian manifold of positive type with regular umbilic boundary and dimension $n \geq 6$. Assume that the Weyl tensor W_g is never vanishing on ∂M. Then, given $\bar{p} > 1$, there exists a positive constant C such that, for any $p \in [\bar{p}, \frac{n}{n-2}]$ and for any $u > 0$, solution of (3.1),*

$$C^{-1} \leq u \leq C \text{ and } \|u\|_{C^{2,\alpha}(M)} \leq C$$

for some $0 < \alpha < 1$.

Theorems 3.3 and 3.5 share the same strategy, inspired by compactness results for classical Yamabe problem. A key point is to give a sharp precise description of a solution around a blow up point. The usual approximation of a rescaled solution is by the standard bubble defined as $V(x) = (1+|x|)^{\frac{2-n}{2}}$. Yamabe boundary problems are no exception, in fact it can be proved that if there exists a sequence $\{u_n\}_n$ of solutions of (2.2), and a sequence $x_n \to x_0$ such that $u_n(x_n) \to +\infty$, then $x_0 \in \partial M$ and u_n, read on $\mathbb{R}^n_+$ and suitably rescaled, is close to the boundary bubble

$$U := \frac{1}{(|\bar{y}|^2 + (1 + y_n^2))^{\frac{n-2}{2}}} \tag{3.2}$$

where $y = (\bar{y}, y_n) \in \mathbb{R}^n_+$, $\bar{y} \in \mathbb{R}^n$ and $y_n \geq 0$.

Unfortunately, this information is not sufficiently accurate to get a result. It is necessary to find a sharp correction term which allows a better description of the solution near a blow up point. This correction term depends on the expansion of the metric in Fermi normal coordinates around $x_0 \in \partial M$, and so on the geometry of the boundary near x_0. In the case of non umbilic boundary, the correction term is a function γ which solves the following linear problem

$$\begin{aligned} -\Delta\gamma &= 2h_{ij}(q)t\partial^2_{ij}U \quad \text{on } \mathbb{R}^n_+; \\ \frac{\partial\gamma}{\partial y_n} &= -nU^{\frac{2}{n-2}}\gamma \quad \text{on } \partial\mathbb{R}^n_+. \end{aligned} \tag{3.3}$$

while for umbilic boundary γ solves

$$\begin{aligned} -\Delta\gamma &= \big[\frac{1}{3}\bar{R}_{ijkl}(q)y_k y_l + R_{ninj}(q)y_n^2\big]\partial^2_{ij}U \quad \text{on } \mathbb{R}^n_+ \\ \frac{\partial\gamma}{\partial y_n} &= -nU^{\frac{2}{n-2}}\gamma \quad \text{on } \partial\mathbb{R}^n_+ \end{aligned} \tag{3.4}$$

where U is the standard bubble for boundary problems defined in (3.2) and, as claimed before, the right hand side of the equation depends on the expansion of the metric near x_0 and keep track of the geometry of the boundary. Here h_{ij} is the trace free second fundamental form, R_{ijkl} is the curvature tensor, and $\bar{R}_{ikjl}$ is the curvature tensor related to the sub-manifold ∂M.

This description, combined with a Pohozaev type inequality, states that in a blow up point the trace free second fundamental form, for non umbilic boundaries, or the Weyl tensor, for umbilic boundaries, must vanish. This allows to prove the compactness theorems.

As far as we known the compactness of solutions when the boundary has both umbilic and non umbilic point has not been considered yet. One reason could be the following: Umbilic and non umbilic blow up points gave rise to different sharp correction terms, and a combined approach which unifies these two cases is not easy to perform.

For umbilic boundaries, the case of dimension $n = 5$ when the Weyl tensor is non vanishing is still open. Dealing with low dimension requires a very precise knowledge of the correction term γ defined in (3.4). For dimensions $n = 6, 7$, the estimates of [16] are enough to get the result. For $n = 5$ a more refined analysis is needed.

One final remark. For non umbilic boundary manifold, compactness holds for all dimensions, by Theorems 3.3 and 3.1, while for umbilic boundary manifold, there exists a non compactness result for dimension $n \geq 25$ ([3]), analogous to what happens in boundaryless manifolds ([7]). At this point one could ask if there is the same similarity between Yamabe problem on manifolds without boundary and Yamabe problem on scalar flat umbilic boundary also for which competes compactness results. In other words, it would be interesting to know if, for dimensions $n \leq 24$, compactness holds on manifolds not locally conformally flat with umbilic boundary, with -at most- the only assumption of positive mass theorem.

4. Stability

Another interesting point is the question of stability: one can ask whether or not the compactness is still true under perturbation of the problem. For Yamabe classical problem Druet, Hebey and Robert in a series of papers [8, 9, 10] studied the stability of the problem under perturbation of the scalar curvature. In particular, they proved that the set of solutions of $-\Delta_g u + \frac{n-2}{4(n-1)}a(x)u = cu^{\frac{n+2}{n-2}}$ in M is still compact if $a(x) \leq R_g(x)$ on M, so problem is *stable* with respect of perturbation of scalar curvature from below. On the other hand, they found counterexamples to compactness, and so *instability,* when $a(x)$ is greater than $R_g(x)$.

The problem of the stability is equivalent to having or not uniform a priori $C^{2,\alpha}$ estimates for the solutions of perturbed problem.

In the same spirit, the authors and Angela Pistoia [15, 21, 22, 17, 20] studied the compactness of the set of solutions to the Yamabe boundary problem under perturbations of the mean and of the scalar curvature. In other words, given $\alpha, \beta : M \to \mathbb{R}$ smooth functions, we study the perturbed problem

$$\begin{gathered} -\Delta_g u + \frac{n-2}{4(n-1)}R_g u + \varepsilon_1 \alpha u = 0 \quad \text{in } M \\ \frac{\partial u}{\partial \nu} + \frac{n-2}{2}h_g u + \varepsilon_2 \beta u = (n-2)u^{\frac{n}{n-2}} \quad \text{on } \partial M \end{gathered} \tag{4.1}$$

where $\varepsilon_1, \varepsilon_2$ are small positive parameters, and we ask if the set of solutions has $C^{2,\alpha}$ uniform bounds, or if there exists a sequence $(\varepsilon_1, \varepsilon_2)$ for which the sequence of solutions $\{u_{\varepsilon_1,\varepsilon_2}\}$ blows up. Again, we studied the case [15, 20, 22] of manifold with umbilic boundary and non umbilic boundary. For umbilic boundary manifolds, we have the following results.

Theorem 4.1. *Let (M, g) be a smooth, n-dimensional Riemannian manifold of positive type not conformally equivalent to the standard ball with regular umbilic boundary ∂M. Let $\alpha, \beta : M \to \mathbb{R}$ be smooth functions such that $\alpha, \beta < 0$ on ∂M. Suppose that $n \geq 8$ and that the Weyl tensor W_g is not vanishing on ∂M. Then, there exists a positive constant C, $0 < \bar{\varepsilon} < 1$ such that, for any $0 \leq \varepsilon_1, \varepsilon_2 \leq \bar{\varepsilon}$ and for any $u > 0$, solution of (4.1),*

$$C^{-1} \leq u \leq C \text{ and } \|u\|_{C^{2,\eta}(M)} \leq C$$

for some $0 < \eta < 1$. The constant C does not depend on $u, \varepsilon_1, \varepsilon_2$.

Theorem 4.2. *Let (M, g) be a smooth, n-dimensional Riemannian manifold of positive type not conformally equivalent to the standard ball with regular umbilic boundary ∂M. Let $\alpha, \beta : M \to \mathbb{R}$ be smooth functions. Suppose that $n \geq 8$ and that the Weyl tensor W_g is not vanishing on ∂M. If $\alpha > 0$ on ∂M or $\beta > 0$ on ∂M, then there exists a sequence of solutions $u_{\varepsilon_1,\varepsilon_2}$ of (4.1) which blows up at a point of the boundary when $(\varepsilon_1, \varepsilon_2) \to (0, 0)$.*

Note that these results are analogous to what happens for classical Yamabe problem. Indeed, if we perturb *from below* one or both the two curvatures, the set of solution is still compact, while it is enough to perturb one of the curvature *from above* to lose compactness. In [20, Remark 27] examples of sign changing α or β for which the solutions blow up are provided.

Quite surprisingly, the analogy between stability of classical and boundary Yamabe problem is lost when dealing with non umbilic boundary manifolds. We have the following result [15, 17, 21]

Theorem 4.3. *Let (M, g) be a smooth, n-dimensional Riemannian manifold of positive type with regular boundary ∂M. Suppose that $n \geq 7$ and that $\pi(x)$, the trace* *free second fundamental form of ∂M, is non zero everywhere. Let $\alpha, \beta : M \to \mathbb{R}$ be smooth functions such that $\beta < 0$ on ∂M and*

$$\max_{q\in\partial M} \left\{\alpha(q) - \frac{n-6}{4(n-1)(n-2)^2}\|\pi(q)\|^2\right\} < 0.$$

Then, there exist two constants $C > 0$ and $0 < \bar{\varepsilon} < 1$ such that, for any $0 \le \varepsilon_1, \varepsilon_2 \le \bar{\varepsilon}$ and for any $u > 0$ solution of (4.1), it holds

$$C^{-1} \le u \le C \text{ and } \|u\|_{C^{2,\eta}(M)} \le C$$

for some $0 < \eta < 1$. The constant C does not depend on $u, \varepsilon_1, \varepsilon_2$.

Theorem 4.4. *Let (M, g) a smooth, n-dimensional Riemannian manifold of positive type with regular boundary ∂M. Suppose that $n \ge 7$ and that the trace free second fundamental form of ∂M, is non zero everywhere. Let $\alpha, \beta : M \to \mathbb{R}$ be smooth functions.*

- *If $\beta > 0$ on ∂M then for $\varepsilon_1, \varepsilon_2 > 0$ small enough there exists a sequence of solutions $u_{\varepsilon_1,\varepsilon_2}$ of (4.1) which blows up at a suitable point of ∂M as $(\varepsilon_1, \varepsilon_2) \to (0, 0)$.*
- *If $\beta < 0$ on ∂M, $\varepsilon_1 = 1$, $\alpha > 0$ on M and $\inf_{q\in\partial M} \alpha(q) + \frac{1}{B}\varphi(q) > 0$, then for $\varepsilon_2 > 0$ small enough there exists a sequence of solutions u_{ε_2} of (4.1) which blows up at a suitable point of ∂M as $\varepsilon_2 \to 0$.*

Here B is a positive constant which depends only on the dimension of M and $\varphi(q) := \frac{1}{2}\int_{\mathbb{R}^n_+} \gamma_q\Delta\gamma_q dy - C\|\pi(q)\|^2 \le 0$, where C is also a positive constant depending only on the dimension.

Theorem 4.3 states that it is possible to have compactness also for perturbation *from above* of the scalar curvature, as long as the perturbation is not to big with respect of the trace free second fundamental form. This was somewhat unexpected, and deeply related to the non umbilicity of the boundary. In fact it turns out that the second fundamental form and the perturbation share the same order of magnitude in the estimates. This makes possible to push a little bit the perturbation above the scalar curvature, compensating it with the second fundamental form.

5. Supercritical case

All compactness theorems for Yamabe boundary problem in Section 3 hold when the exponent p in (3.1) belongs to an interval $[\bar{p}, \frac{n}{n-2}]$ for some $\bar{p} > 1$. These results can be also interpreted in terms of stability: we can say that the Yamabe boundary problem (2.1) is *stable* for perturbation of the critical exponent *from below*. At this point it is natural to ask wether if problem (2.1) is stable for perturbation of the critical exponent from above. In fact, in [18, 19] the problem

$$\begin{gathered} -\Delta_g u + \frac{n-2}{4(n-1)}R_g u = 0 \quad \text{in } M \\ \frac{\partial u}{\partial \nu} + \frac{n-2}{2}h_g u = (n-2)u^{\frac{n}{n-2}+\varepsilon} \quad \text{on } \partial M \end{gathered} \tag{5.1}$$

is studied for $\varepsilon > 0$, and the authors give the following negative answer.

Theorem 5.1. *Let M be a manifold of positive type with boundary, and let $\varepsilon > 0$.*

- *If M has non umbilic boundary, assume $n \ge 7$*

- *In M has umbilic boundary, assume $n \geq 8$ and that the Weyl tensor is not vanishing on ∂M.*

Then there exists a family of solutions u_ε of (5.1) *which blows up at a point on ∂M as $\varepsilon \to 0^+$.*

The proof of Theorem 5.1 relies on the Lyapunov-Schmidt reduction. In particular, for any -sufficiently small- positive ε, it is possible to construct a solution u_ε of (5.1) which is a sum of a boundary bubble, the sharp correction term defined in (3.3) or in (3.4), both suitably rescaled and centered in a point on ∂M, and a small remainder term. The function u_ε blows up since the rescaling forces the maximum value of the bubble to diverge. As a byproduct, these proofs also construct an almost explicit solution to the supercritical Yamabe boundary problem, and supercritical solution are, in general, more difficult to find than their subcritical counterparts. Theorem 5.1, combined with the theorems in section 3, states that, given $\bar{p} \in (1, \frac{n}{n-2})$ and $\bar{\varepsilon} > 0$ sufficiently small, there exist solutions u_p of (2.2) for all $p \in [\bar{p}, \frac{n}{n-2} + \bar{\varepsilon}]$. In addiction, for $p \leq \frac{n}{n-2}$, u_p belongs to $C^2(M)$ and $\|u_p\|_{C^2} \leq C$, for a positive constant C not depending on p and u_p, while, for $p > \frac{n}{n-2}$, u_p belongs to $H^1 \cap L^{\frac{2(n-1)}{n-2} + n\varepsilon}$, where $\varepsilon = p - \frac{n}{n-2}$, but the L^∞ norm of u_p could blow up as p approaches $\frac{n}{n-2}$ from above.

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