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INITIAL VALUE PROBLEMS FOR NONLINEAR NONRESONANT DELAY DIFFERENTIAL EQUATIONS WITH POSSIBLY INFINITE DELAY

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ABSTRACT. We study initial value problems for scalar, nonlinear, delay differential equations with distributed, possibly infinite, delays. We consider the initial value problem

$$\begin{cases} x(t) = \varphi(t), & t \le 0 \\ x'(t) + \displaystyle\int_0^\infty g(t, s, x(t), x(t-s))\, d\mu(s) = f(t), & t \ge 0, \end{cases}$$

where φ and f are bounded and μ is a finite Borel measure. Motivated by the nonresonance condition for the linear case and previous work of the authors, we introduce conditions on g. Under these conditions, we prove an existence and uniqueness theorem. We show that under the same conditions, the solutions are globally asymptotically stable and, if μ satisfies an exponential decay condition, globally exponentially asymptotically stable.

1. INTRODUCTION

In this paper we will study the initial value problem for scalar, nonlinear, delay differential equations with possibly infinite delay. We will consider problems of the form

$$\begin{cases} x(t) = \varphi(t), & t \le 0 \\ x'(t) + \displaystyle\int_0^\infty g(t, s, x(t), x(t-s))\, d\mu(s) = f(t), & t \ge 0. \end{cases} \tag{1.1}$$

We assume that φ is bounded and continuous on $(-\infty, 0]$, f is bounded and continuous on $[0, \infty)$, and that μ is a positive, finite Borel measure on $[0, \infty)$. As usual, $x'(0)$ in (1.1) is to be interpreted as a right-hand derivative. In all the cases we consider, g will be continuous.

In this paper, we will give conditions on g that will ensure that the initial value problem (1.1) has a unique maximally defined solution, which is defined on the

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entire real line $\mathbb{R}$. We will show that the same conditions on g ensure that the solutions of (1.1) are asymptotically stable, i.e., if x_1 and x_2 are solutions of (1.1) for different initial conditions φ_1 and φ_2 (but with the same forcing function f), then $x_1(t)-x_2(t)\to 0$ as $t\to\infty$. We will also show that if (1.1) has exponentially fading memory, i.e., μ decays exponentially, then every solution of (1.1) is exponentially asymptotically stable, i.e., if x_1 and x_2 are solutions of (1.1) for different initial conditions, then

$$|x_1(t)-x_2(t)|\le Ce^{-\lambda t},\qquad t\ge 0,$$

for some constants C and $\lambda>0$.

In the rest of this introduction, we will describe the conditions that we will place on g. To motivate these conditions, it will help to recall some previous work of the authors in [7] (for related work of the authors see [3, 4, 5, 6, 8, 12]).

In [7], the authors studied delay differential equations of the form

$$x'(t)+\int_{-\infty}^{\infty} g(x(t),x(t-s))\,d\mu(s)=f(t),\qquad t\in\mathbb{R}\,, \tag{1.2}$$

under a generalized nonresonance condition Condition GNR. Under this condition, it was shown that (1.2) has a unique solution that is defined and bounded on all of $\mathbb{R}$. We want to briefly recall the statement and motivation of Condition GNR.

A very special case of (1.2) is the linear constant coefficient equation

$$x'(t)+ax(t)+bx(t-\tau)=0, \tag{1.3}$$

where $a,b\in\mathbb{R}$. This equation can be analyzed by classical techniques, [1, 9, 10]. In particular, (1.3) has a nontrivial bounded solution if and only if the characteristic equation $z+a+be^{-\tau z}=0$ has a root on the imaginary axis. Thus, it can be shown that the set C_τ of pairs (a,b) in the ab-plane for which (1.3) has a nontrivial bounded solution consists of the line $a+b=0$ and the multi-branch parameterized curve

$$(a,b)=\frac{1}{\tau}(-\theta\cot(\theta),\theta\csc(\theta)).$$

It is known that if (a,b) lies to the right of C_τ the zero solution of (1.3) is globally asymptotically stable (i.e., all of the roots of the characteristic equation are in the left half plane), see [9, 10, 11]. As τ varies, C_τ will sweep out the region R' consisting of the two quadrants above and below the lines $a\pm b=0$, with $a+b=0$ included and $a-b=0$ excluded. See Figure 1, which shows a few branches of C_τ and shows the line $a-b=0$ as dotted.

Equation (1.3) is a special case of (1.2) with $g(x,y)=ax+by$ (μ is the Dirac measure at τ). In this case, $\{\,(a,b)\,\}$ is the image of the gradient of g, ∇g. In the nonlinear case, the image of ∇g will be more than a single point. To get results for all delays, we want the image of ∇g to avoid R'. As the *first part of Condition GNR*, we required the somewhat stronger condition that the image of ∇g be disjoint from

$$R=\left\{\,(a,b)\in\mathbb{R}^2\mid |a|\le|b|\,\right\},$$

the closure of R' (in this paper, the image will have to lie to the right of R).

It is possible that $\operatorname{Im}(\nabla g)$, the image of ∇g, comes arbitrarily close to R. We need some control on how fast $\operatorname{Im}(\nabla g)$ approaches R. This is measured as follows. For $\rho\ge 0$, define

$$Q(\rho)=\left\{\,(x,y)\in\mathbb{R}^2\mid |x|\le\rho,|y|\le\rho\,\right\}.$$

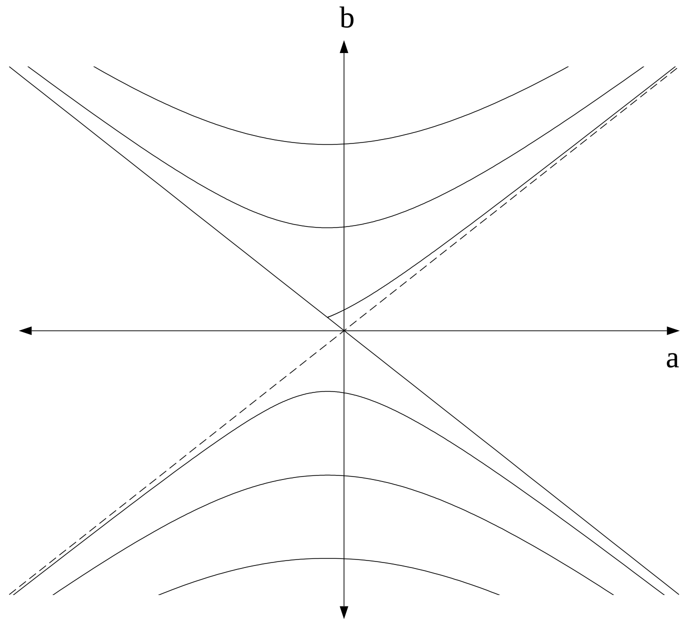

FIGURE 1. The set C_τ

and let $G(\rho) = \nabla g(Q(\rho))$. Let $\alpha, \beta \colon \mathbb{R}^2 \to \mathbb{R}$ be the linear functionals

$$\begin{aligned} \alpha(a, b) &= a - b \\ \beta(a, b) &= a + b. \end{aligned}$$

The boundary lines of R are $\alpha = 0$ and $\beta = 0$. The region to the right of R is described by $\alpha > 0$ and $\beta > 0$, while the region to the left of R is described by $\alpha < 0$ and $\beta < 0$. For $\rho \geq 0$ we define

$$\begin{aligned} \alpha_*(\rho) &= \inf \{ \alpha(a, b) \mid (a, b) \in G(\rho) \} \\ \alpha^*(\rho) &= \sup \{ \alpha(a, b) \mid (a, b) \in G(\rho) \} \\ \beta_*(\rho) &= \inf \{ \beta(a, b) \mid (a, b) \in G(\rho) \} \\ \beta^*(\rho) &= \sup \{ \beta(a, b) \mid (a, b) \in G(\rho) \} . \end{aligned} \tag{1.4}$$

Consider the case where $\mathrm{Im}(\nabla g)$ lies to the right of R. In this case, we define

$$\begin{aligned} r(\rho) &= \min \{ \alpha_*(\rho), \beta_*(\rho) \} \\ s(\rho) &= \max \{ \alpha^*(\rho), \beta^*(\rho) \} . \end{aligned} \tag{1.5}$$

See Figure 2 for an illustration. Clearly r is a positive non-increasing function. If $\mathrm{Im}(\nabla g)$ comes arbitrarily close to R, we will have $r(\rho) \to 0$ as $\rho \to \infty$, and the rate at which r goes to zero is a measure of how fast $\mathrm{Im}(\nabla g)$ approaches R. As the *second part of Condition GNR*, we assume that

$$\sup \{ \rho r(\rho) \mid \rho \geq 0 \} = \infty.$$

Similar definitions can be made in the case where $\operatorname{Im}(\nabla g)$ lies to the left of R, but these will not be needed in this paper. We do not need to impose any assumptions on $s(\rho)$, but it will figure in our proofs.

We want to extend Condition GNR to allow g to depend explicitly on t and s, as in (1.1). This is necessary for the techniques we will use in analyzing the initial value problem, as well as desirable for greater generality. For brevity, we will refer to the case where g does not depend explicitly on t and s as the "time independent case."

Our extended condition also takes into account another consideration. Since the method of steps does not apply to (1.1), it is not clear that we have unique continuation of solutions for (1.1). In order to prove uniqueness, it will be necessary to consider solutions defined on intervals with a finite upper endpoint.

These considerations lead us to the following definition of the class of functions g we will consider.

Definition 1.1. For $0 < p \leq \infty$, let $\mathcal{G}_p$ denote the set of functions

$$g\colon [0, p) \times [0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\colon (t, s, x, y) \mapsto g(t, s, x, y)$$

that satisfy the following conditions.

(G1) g is continuous.

(G2) The function

$$g(\cdot, \cdot, 0, 0)\colon [0, p) \times [0, \infty) \to \mathbb{R}\colon (t, s) \mapsto g(t, s, 0, 0)$$

is bounded.

(G3) The partial derivatives g_x and g_y exist and are continuous.

Since we will not have occasion to differentiate g with respect to t or s, we will use the notation ∇g for the function $(t, s, x, y) \mapsto (g_x(t, s, x, y), g_y(t, s, x, y))$.

(G4) For every compact set $K \subseteq \mathbb{R}^2$, the image of $[0, p) \times [0, \infty) \times K$ under ∇g is a bounded set whose closure is disjoint from R and lies to the right of R.

For $\rho \geq 0$, let $G(\rho)$ be the image of $[0, p) \times [0, \infty) \times Q(\rho)$ under ∇g. Let $\alpha_*(\rho)$, $\alpha^*(\rho)$, $\beta_*(\rho)$, $\beta^*(\rho)$, $r(\rho)$ and $s(\rho)$ be defined as in (1.4) and (1.5). Our last condition is the following.

(G5) g satisfies

$$\sup \{ \rho r(\rho) \mid \rho \geq 0 \} = \infty.$$

A time independent g that satisfies Condition GNR (and has $\operatorname{Im}(\nabla g)$ lying to the right of R) is a member of $\mathcal{G}_p$ for all p. Further examples can be generated using the $\mathcal{G}$-lemma of the next section.

In the next section we will derive the basic properties of $\mathcal{G}_p$. In Section 3, we will prove the existence and uniqueness of a solution to our initial value problem. In Section 4, we prove the solutions are globally asymptotically stable. In Section 5, we prove the solutions are exponentially asymptotically stable under an exponential decay assumption on μ.

2. Basic estimates

In this section, we will establish the basic properties of functions in $\mathcal{G}_p$ that we will use. The first thing we need is an elementary geometric estimate, see [7].

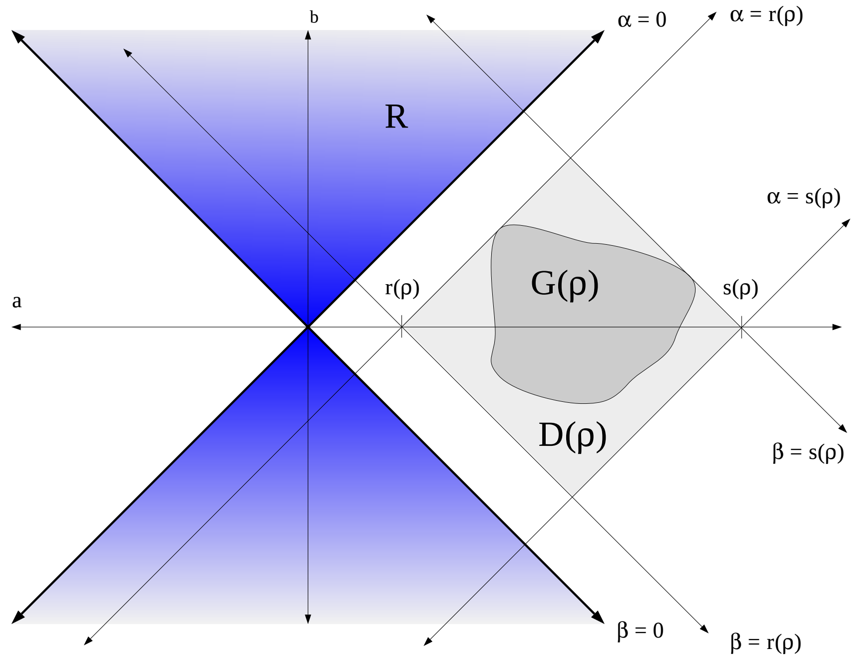

FIGURE 2. The setting for Lemma 2.2.

Lemma 2.1. *Let D be the region in the ab-plane defined by the inequalities*

$$r \leq \alpha, \beta \leq s$$

for constants r and s. Let a be a fixed real number. Then we have

$$\sup \{ |a-h| + |k| \mid (h,k) \in D \} = \max \{ |a-r|, |a-s| \} . \tag{2.1}$$

To picture the region D, see $D(\rho)$ in Figure 2. The following lemma gives the basic estimate for functions of class $\mathcal{G}_p$.

Lemma 2.2. *Suppose that $g \in \mathcal{G}_p$ and let a be a fixed real number. If $\rho \geq 0$ and $(\xi_i, \eta_i) \in Q(\rho)$, $i = 1, 2$, we have the estimate*

$$\begin{aligned} &|[a\xi_1 - g(t,s,\xi_1,\eta_1) - [a\xi_2 - g(t,s,\xi_2,\eta_2)]| \\ &\qquad \leq K(a,\rho) \max \{ |\xi_1 - \xi_2|, |\eta_1 - \eta_2| \} , \end{aligned} \tag{2.2}$$

for all $(t,s) \in [0,p) \times [0,\infty)$. Here $K(a,\rho)$ is defined as

$$K(a,\rho) = \max \{ |a - r(\rho)|, |a - s(\rho)| \} .$$

From this, we get the estimate

$$|g(t,s,\xi,\eta)| \leq K(0,\rho) \max \{ |\xi|, |\eta| \} + |g(t,s,0,0)|, \qquad (\xi,\eta) \in Q(\rho). \tag{2.3}$$

See Figure 2 for the setting of this lemma.

Proof. Consider the norms defined on $\mathbb{R}^2$ by

$$\|(a,b)\|_1 = |a| + |b|$$
$$\|(a,b)\|_\infty = \max \{ |a|, |b| \} .$$

Let $u_i = (\xi_i, \eta_i)$, $i = 1, 2$. Let $t_0 \in [0, p)$ and $s_0 \in [0, \infty)$ be fixed but arbitrary, and define $\varphi(\xi, \eta) = a\xi - g(t_0, s_0, \xi, \eta)$. This function is continuously differentiable, and by applying the Mean Value Theorem to $\sigma \mapsto \varphi((1-\sigma)u_1 + \sigma u_2)$, we obtain

$$\varphi(u_2) - \varphi(u_1) = \nabla\varphi(u^*) \cdot (u_2 - u_1), \tag{2.4}$$

for some point $u^* = (\xi^*, \eta^*)$ on the line segment joining u_1 and u_2. Since $Q(\rho)$ is convex, $u^* \in Q(\rho)$. From (2.4), we have

$$|\varphi(u_2) - \varphi(u_1)| \leq \|\nabla\varphi(u^*)\|_1 \, \|u_2 - u_1\|_\infty. \tag{2.5}$$

Let $D(\rho)$ be the region in the ab-plane defined by the inequalities $r(\rho) \leq \alpha, \beta \leq s(\rho)$. We may estimate $\|\nabla\varphi(u^*)\|_1$ as follows:

$$\begin{aligned}
\|\nabla\varphi(u^*)\|_1 &= |a - g_x(t_0, s_0, \xi^*, \eta^*)| + |-g_y(t_0, s_0, \xi^*, \eta^*)| \\
&= |a - g_x(t_0, s_0, \xi^*, \eta^*)| + |g_y(t_0, s_0, \xi^*, \eta^*)| \\
&\leq \sup\{\, |a - g_x(t, s, \xi, \eta)| + |g_y(t, s, \xi, \eta)| \mid \\
&\qquad\qquad (t, s, \xi, \eta) \in [0, p) \times [0, \infty) \times Q(\rho) \,\} \\
&= \sup\{\, |a - h| + |k| \mid (h, k) \in G(\rho) \,\} \\
&\leq \sup\{\, |a - h| + |k| \mid (h, k) \in D(\rho) \,\} \\
&= \max\{\, |a - r(\rho)|, |a - s(\rho)| \,\},
\end{aligned}$$

using Lemma 2.1. Using the definition of $K(a, \rho)$ and (2.5), we have

$$|\varphi(u_1) - \varphi(u_1)| \leq K(a, \rho) \|u_2 - u_1\|_\infty,$$

which translates into (2.2) when the definitions are expanded. The inequality in (2.3) comes from (2.2) by setting $a = 0$, $(\xi_1, \eta_1) = (0, 0)$ and $(\xi_2, \eta_2) = (\xi, \eta)$, and using the triangle inequality. $\square$

For a topological space X, we will use $BC(X)$ to denote the space of bounded continuous functions $f \colon X \to \mathbb{R}$, equipped with the supremum norm, which will be denoted by $\|f\|$. If X is an interval, we omit the outer parentheses.

The next two lemmas show that the class $\mathcal{G}_p$ is closed under an operation that will be frequently employed in our proofs.

Lemma 2.3. *Suppose that $g \in \mathcal{G}_p$ and that x and y are bounded continuous functions on $[0, p) \times [0, \infty)$. Then the function*

$$(t, s) \mapsto g(t, s, x(t, s), y(t, s))$$

is bounded on $[0, p) \times [0, \infty)$.

Proof. Choose ρ such that $\|x\|, \|y\| \leq \rho$. Then $(x(t, s), y(t, s)) \in Q(\rho)$ for all $(t, s) \in [0, p) \times [0, \infty)$. Thus, (2.3) shows that

$$|g(t, s, x(t, s), y(t, s))| \leq K(0, \rho) \max\{\, |x(t, s)|, |y(t, s)| \,\} + |g(t, s, 0, 0)|.$$

By hypothesis G2, the function $g(\cdot, \cdot, 0, 0)$ is bounded, and so

$$|g(t, s, x(t, s), y(t, s))| \leq K(0, \rho)\rho + \|g(\cdot, \cdot, 0, 0)\|.$$

$\square$

Lemma 2.4 (The $\mathcal{G}$-lemma)**.** *Suppose that $g \in \mathcal{G}_p$ and suppose that*

$$x, y, f \in BC([0,p) \times [0,\infty)).$$

Let $h\colon [0,p) \times [0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$h(t,s,\xi,\eta) = g(t,s,\xi + x(t,s), \eta + y(t,s)) + f(t,s).$$

Then h is in $\mathcal{G}_p$.

Proof. The function h is clearly continuous, so hypothesis G1 is satisfied. The last lemma shows that $h(\cdot,\cdot,0,0)$ is bounded, so G2 is satisfied. The gradient of h is given by

$$\nabla h(t,s,\xi,\eta) = (g_x(t,s,\xi + x(t,s), \eta + y(t,s)), g_y(t,s,\xi + x(t,s), \eta + y(t,s)),$$

and G3 is satisfied.

Choose $M > 0$ such that $\|x\|, \|y\| \leq M$. Then, if $(\xi,\eta) \in Q(\rho)$, the point $(\xi + x(t,s), \eta + y(t,s))$ is in $Q(\rho + M)$ for all (t,s). Thus, if we let $H(\rho)$ denote the image of $[0,p) \times [0,\infty) \times Q(\rho)$ under ∇h, we have

$$H(\rho) \subseteq G(\rho + M). \tag{2.6}$$

Thus, $H(\rho)$ is a bounded set whose closure lies to the right of R, so G4 is satisfied.

If we let r_h and r_g denote the r-functions for h and g respectively, (2.6) shows that $r_g(\rho + M) \leq r_h(\rho)$. Thus, to show that $\sup \rho r_h(\rho) = \infty$, it will suffice to show that $\sup \rho r_g(\rho + M) = \infty$. To prove this, let $A > 0$ be arbitrary. Since $\sup \rho r_g(\rho) = \infty$, we can find some σ such that $\sigma r_g(\sigma) \geq A + M r_g(0)$. We must have $\sigma > M$, for otherwise $\sigma r_g(\sigma) \leq M r_g(0)$, since r_g is non-increasing. Thus, we can write $\sigma = \rho + M$ for $\rho > 0$. We have

$$(\rho + M) r_g(\rho + M) \geq A + M r_g(0)$$

which implies that

$$\rho r_g(\rho + M) \geq A + M[r_g(0) - r_g(\rho + M)].$$

The right hand side is greater that or equal to A, since r_g is non-increasing. Since A was arbitrary, we conclude that $\sup \rho r_g(\rho + M) = \infty$. Thus, h satisfies G5. $\square$

We should also observe the following lemma, whose proof is straight forward.

Lemma 2.5. *Suppose that $g \in \mathcal{G}_p$ and that $0 < q < p$. Then, the restriction of g to $[0,q) \times [0,\infty) \times \mathbb{R}^2$ is in $\mathcal{G}_q$.*

3. Existence and Uniqueness

In this section we prove the existence and uniqueness of solutions of the initial value problem (1.1). Our first goal is the following Proposition.

Proposition 3.1. *Suppose that $g \in \mathcal{G}_p$ and that μ is a finite positive Borel measure on $[0,\infty)$. Then, for every $\varphi \in BC(-\infty,0]$ and $f \in BC[0,\infty)$, there is a unique function $x \in BC(-\infty,p)$ that satisfies the initial value problem*

$$\begin{cases} x(t) = \varphi(t), & t \in (-\infty,0] \\ x'(t) + \displaystyle\int_0^\infty g(t,s,x(t),x(t-s))\,d\mu(s) = f(t), & t \in [0,p). \end{cases} \tag{3.1}$$

It will be useful to observe that if x is a bounded solution of (3.1), then x' must be continuous and bounded on $[0, \infty)$.

The first step in proving this proposition is to observe that we can reduce the problem to the case where the initial condition φ is zero.

To see this, let $\bar{\varphi}$ be an extension of φ to the interval $(-\infty, p)$, such that $\bar{\varphi}$ is bounded, $\bar{\varphi}$ is C^1 on $[0, p)$ (interpreting $\bar{\varphi}'(0)$ as a right hand derivative, as usual) and $\bar{\varphi}'$ is bounded on $[0, p)$. An obvious choice would be to define $\bar{\varphi}(t) = \varphi(0)$ for $t > 0$, but another choice will be useful in our stability analysis.

Suppose that x is a solution of the initial value problem (3.1). Define $y = x - \bar{\varphi}$. Then y is bounded and continuous and y is continuously differentiable on $[0, p)$. Of course, $y(t) = 0$ for $t \in (-\infty, 0]$. If we substitute $x = y + \bar{\varphi}$ in the initial value problem, for $t \in [0, p)$ we obtain

$$y'(t) + \bar{\varphi}'(t) + \int_0^\infty g(t, s, y(t) + \bar{\varphi}(t), y(t-s) + \bar{\varphi}(t-s))\, d\mu(s) = f(t)\,. \tag{3.2}$$

Let $m = \mu[0, \infty)$ denote the total mass of μ, a notation that will be used for the rest of the paper. Then, for $t \in [0, p)$ we can rewrite (3.2) as

$$y'(t) + \int_0^\infty [g(t, s, y(t) + \bar{\varphi}(t), y(t-s) + \bar{\varphi}(t-s)) + \bar{\varphi}'(t)/m]\, d\mu(s) = f(t)\,.$$

Thus, if we define h by

$$h(t, s, \xi, \eta) = g(t, s, \xi + \bar{\varphi}(t), \eta + \bar{\varphi}(t-s)) + \bar{\varphi}'(t)/m\,,$$

we see that y is a solution of the initial value problem

$$\begin{cases} y(t) = 0, & t \in (-\infty, 0] \\ y'(t) + \displaystyle\int_0^\infty h(t, s, y(t), y(t-s))\, d\mu(s) = f(t), & t \in [0, p). \end{cases} \tag{3.3}$$

By the $\mathcal{G}$-lemma, h is again in $\mathcal{G}_p$.

Conversely, if y is a bounded solution of (3.3), then $x = y + \bar{\varphi}$ is a bounded solution of (3.1). Thus, to prove Proposition 3.1, it will suffice to prove the following Proposition.

Proposition 3.2. *Suppose that $g \in \mathcal{G}_p$ and that μ is a finite positive Borel measure on $[0, \infty)$. Then, for every $f \in BC[0, \infty)$, there is a unique function $x \in BC(-\infty, p)$ that satisfies the initial value problem*

$$\begin{cases} x(t) = 0, & t \in (-\infty, 0] \\ x'(t) + \displaystyle\int_0^\infty g(t, s, x(t), x(t-s))\, d\mu(s) = f(t), & t \in [0, p). \end{cases} \tag{3.4}$$

The proof of this Proposition will occupy most of the rest of this section. We begin by introducing the function spaces we will use. Let $BC^1[0, p)$ denote the space of functions $f \in BC[0, p)$ such that f is differentiable and $f' \in BC[0, p)$.

Let X_p be the space of functions $x \in BC(-\infty, p)$ such that $x = 0$ on the negative half-axis $(-\infty, 0]$. This is a closed subspace of $BC(-\infty, p)$ and hence a Banach space in the supremum norm. Finally, let X_p^1 denote the space of functions $x \in X_p$ such that the restriction of x to $[0, p)$ is in $BC^1[0, p)$.

The simplest case of the initial value problem (3.4) is the problem

$$\begin{cases} x(t) = 0, & t \in (-\infty, 0] \\ x'(t) + ax(t) = f(t), & t \in [0, p), \end{cases}$$

where $a > 0$. This is, of course, easy to solve by elementary means. The results are summarized in the next lemma.

Lemma 3.3. *For $a > 0$, let $L_a \colon X_p^1 \to BC[0, p)$ be the operator defined by*

$$L_a x(t) = x'(t) + ax(t), \qquad t \in [0, p).$$

Then, L_a is invertible, with the inverse given by

$$L_a^{-1} f(t) = \begin{cases} 0, & t \in (-\infty, 0] \\ \displaystyle\int_0^t e^{a(s-t)} f(s)\, ds, & t \in [0, p). \end{cases}$$

From this formula we get the supremum norm estimate

$$\|L_a^{-1} f\| \le \frac{1}{a} \|f\|. \tag{3.5}$$

We will now introduce some operators that will be useful in the proof. If $a > 0$ and $x \in X_p$, define a function $N_a(x)$ on $[0, p) \times [0, \infty)$ by

$$N_a(x)(t, s) = ax(t) - g(t, s, x(t), x(t - s)).$$

By Lemma 2.3, this function is bounded, so we have a nonlinear operator $N_a \colon X_p \to BC([0, p) \times [0, \infty))$.

We will use the notation $B(\rho)$ for the closed ball of radius ρ centered at the origin in X_p. If $x \in B(\rho)$, then $(x(t), x(t - s)) \in Q(\rho)$ for all t and s. Thus, if $x, y \in B(\rho)$ and we apply Lemma (2.2), we have

$$\begin{aligned} &|[ax(t) - g(t, s, x(t), x(t - s))] - [ay(t) - g(t, s, y(t), y(t - s))]| \\ &\quad \le \; K(a, \rho) \max\{\, |x(t) - y(t)|, |x(t - s) - y(t - s)| \,\} \\ &\quad \le \; K(a, \rho)\|x - y\|, \end{aligned}$$

since both terms in the maximum are bounded by $\|x - y\|$. Thus, we have the estimate

$$\|N_a(x) - N_a(y)\| \le K(a, \rho)\|x - y\|, \qquad x, y \in B(\rho). \tag{3.6}$$

For $a > 0$, we define a nonlinear operator $M_a \colon X_p \to BC[0, p)$ by

$$M_a(x)(t) = ax(t) - \int_0^\infty g(t, s, x(t), x(t - s))\, d\mu(s), \qquad t \in [0, p).$$

Recalling the notation $m = \mu[0, \infty)$, we can rewrite this as

$$\begin{aligned} M_a(x)(t) &= \int_0^\infty [(a/m)x(t) - g(t, s, x(t), x(t - s))]\, d\mu(s) \\ &= \int_0^\infty N_{a/m}(x)(t, s)\, d\mu(s). \end{aligned}$$

Hence, by applying (3.6) we obtain the estimate

$$\|M_a(x) - M_a(y)\| \le mK(a/m, \rho)\|x - y\|, \qquad x, y \in B(\rho). \tag{3.7}$$

We next show that (3.4) can be reduced to a fixed point problem in X_p. Suppose that $x \in X_p$ is a solution of (3.4). Then, if $a > 0$ is arbitrary, we have, for $t \geq 0$,

$$x'(t) + ax(t) = ax(t) - \int_0^\infty g(t, s, x(t), x(t-s))\, d\mu(s) + f(t)$$

We may rewrite this as

$$L_a x = M_a(x) + f$$

Since L_a is invertible, this is equivalent to

$$x = T_a(x)\,,$$

where $T_a \colon X_p \to X_p$ is defined by

$$T_a(x) = L_a^{-1} M_a(x) + L_a^{-1} f.$$

Conversely, if $x \in X_p$ and $x = T_a(x)$, then $x \in X_p^1 \subseteq X_p$, and we may reverse the steps to conclude that x is a solution of (3.4). Thus, we have the following lemma.

Lemma 3.4. *The following conditions are equivalent.*

1. $x \in X_p$ *is a solution of* (3.4).
2. $x \in X_p$ *and* $T_a(x) = x$ *for* all $a > 0$.
3. $x \in X_p$ *and* $T_a(x) = x$ *for* some $a > 0$.

To show that one of the operators T_a has a fixed point, we will use the Contraction Mapping Lemma. From (3.7) and (3.5), we get the estimate

$$\|T_a(x) - T_a(y)\| \leq \frac{m}{a} K(a/m, \rho) \|x - y\|, \qquad x, y \in B(\rho)\,. \tag{3.8}$$

We now make a specific choice of a. For $\rho \geq 0$, define

$$a(\rho) = m \frac{r(\rho) + s(\rho)}{2}\,.$$

It is then easily calculated that

$$K(a(\rho)/m, \rho) = \frac{s(\rho) - r(\rho)}{2}$$

and thus that

$$\frac{a(\rho)}{m} - K(a(\rho)/m, \rho) = r(\rho) > 0\,. \tag{3.9}$$

In particular, $K(a(\rho)/m, \rho) < a(\rho)/m$, and so

$$\frac{m}{a(\rho)} K(a(\rho)/m, \rho) < 1\,.$$

If we set $a = a(\rho)$ in (3.8), we have

$$\|T_{a(\rho)}(x) - T_{a(\rho)}(y)\| \leq \frac{m}{a(\rho)} K(a(\rho)/m, \rho) \|x - y\|, \qquad x, y \in B(\rho)\,,$$

where the constant is strictly less than one. This is enough to prove the uniqueness part of Proposition (3.2). If $x, y \in X_p$ are solutions of the initial value problem (3.4), we can choose ρ such that $x, y \in B(\rho)$. By Lemma (3.4), x and y are fixed points of $T_{a(\rho)}$ and hence

$$\|x - y\| \leq \frac{m}{a(\rho)} K(a(\rho)/m, \rho) \|x - y\|\,.$$

Since the constant is strictly less than one, this implies $x = y$.

We have not yet proven the existence of a solution, because we don't know that $T_{a(\rho)}$ maps $B(\rho)$ into itself. If we can find a ρ such that $T_{a(\rho)}(B(\rho)) \subseteq B(\rho)$, it will follow from the Contraction Mapping Lemma that $T_{a(\rho)}$ has a fixed point, and hence that there is a solution $x \in X_p$ of the initial value problem (3.4).

To find such a ρ, we make some additional estimates. Setting $y = 0$ in (3.7) gives

$$\|M_a(x)\| \leq mK(a/m, \rho)\|x\| + m\|g(\cdot, \cdot, 0, 0)\|, \qquad x \in B(\rho),$$

For brevity, set $\gamma = \|g(\cdot, \cdot, 0, 0)\|$. By the definition of T_a, we have

$$\|T_a(x)\| \leq \frac{1}{a}\|M_a(x)\| + \frac{1}{a}\|f\|.$$

Thus, we have

$$\|T_a(x)\| \leq \frac{m}{a}K(a/m, \rho)\|x\| + \frac{m}{a}\gamma + \frac{1}{a}\|f\|, \qquad x \in B(\rho).$$

If we set $a = a(\rho)$ in the last inequality and estimate $\|x\|$ by ρ, we obtain the estimate

$$\|T_{a(\rho)}(x)\| \leq \frac{m}{a(\rho)}K(a(\rho)/m, \rho)\rho + \frac{m}{a(\rho)}\gamma + \frac{1}{a(\rho)}\|f\|, \qquad x \in B(\rho).$$

From this inequality, we see that if we can choose ρ such that

$$\frac{m}{a(\rho)}K(a(\rho)/m, \rho)\rho + \frac{m}{a(\rho)}\gamma + \frac{1}{a(\rho)}\|f\| \leq \rho, \tag{3.10}$$

we will have $T_{a(\rho)}(B(\rho)) \subseteq B(\rho)$. If we multiply both sides of (3.10) by $a(\rho)/m$ and move the first term on the left to the other side, we have

$$\gamma + \frac{1}{m}\|f\| \leq \left[\frac{a(\rho)}{m} - K(a(\rho)/m, \rho)\right]\rho\,.$$

By (3.9), this reduces to

$$\gamma + \frac{1}{m}\|f\| \leq \rho r(\rho)\,. \tag{3.11}$$

In this inequality, the left hand side is independent of ρ, while the right hand side can be made as large as we like by our assumption G5. Thus, we can choose ρ to satisfy (3.11).

This completes the proof of Proposition 3.2, and hence the proof of Proposition 3.1.

We can now apply Proposition 3.1 to prove our main existence and uniqueness theorem.

Theorem 3.5. *Suppose that $g \in \mathcal{G}_p$, where $0 < p \leq \infty$, and that μ is a finite positive Borel measure on $[0, \infty)$. Then for every $\varphi \in BC(-\infty, 0]$ and $f \in BC[0, p)$, the initial value problem*

$$\begin{cases} x(t) = \varphi(t), & t \in (-\infty, 0] \\ x'(t) + \displaystyle\int_0^\infty g(t, s, x(t), x(t-s))\, d\mu(s) = f(t), & t \in [0, p) \end{cases} \tag{3.12}$$

has a unique maximally defined solution x, which is defined and bounded on $(-\infty, p)$.

Proof. We know, of course, that there is a solution x of (3.12) which is defined and bounded on $(-\infty, p)$.

To prove the uniqueness assertion, suppose that y is a function, possibly unbounded, defined on $(0, q)$, where $0 < q \le p$, that satisfies (3.12) on its domain. Let $t_0 \in (0, q)$ be fixed but arbitrary. Choose some point u such that $t_0 < u < q$. By continuity, both x and y are bounded on $[0, u]$, and hence on $(-\infty, u)$. The restriction of g to $[0, u) \times [0, \infty) \times \mathbb{R}^2$ is in the class $\mathcal{G}_u$ and the restrictions of x and y to $(-\infty, u)$ are both bounded solutions of the corresponding initial value problem. Thus, by the uniqueness part of Proposition 3.1, we must have $x = y$ on $(-\infty, u)$. In particular, $x(t_0) = y(t_0)$. Since $t_0 \in (0, q)$ was arbitrary, we conclude that that $x = y$ on the intersection of their domains. □

We observe that the argument in the proof can be extended to show the existence of a unique maximally defined solution to (3.12) even in the case where f is unbounded, but we will not pursue this result here.

It's not hard to trace through our proof to show that the choice of ρ and the contraction constant can be made uniformly for φ and f in a closed ball in $BC(-\infty, 0] \times BC[0, p)$, and then to use the Uniform Contraction Principal [2, page 25] to show that the solution x depends continuously on the data (φ, f).

Under an additional assumption on g, which is automatically satisfied in the time independent case and is preserved by the operations in the $\mathcal{G}$-lemma, it can be shown that the dependence of x on (φ, f) is C^1. The details are sufficiently involved that we won't pursue them here.

4. Asymptotic Stability

For the rest of the paper, we will be concerned with the case $p = \infty$ and we will write $\mathcal{G} = \mathcal{G}_p$, $X = X_p$, etc. Our goal in this section is the following theorem.

Theorem 4.1. *Suppose that $g \in \mathcal{G}$ and let μ be a finite positive Borel measure on $[0, \infty)$. Suppose that $f \in BC[0, \infty)$ and $\varphi \in BC(-\infty, 0]$ and consider the initial value problem*

$$\begin{cases} x(t) = \varphi(t), & t \in (-\infty, 0] \\ x'(t) + \displaystyle\int_0^\infty g(t, s, x(t), x(t-s))\, d\mu(s) = f(t), & t \in [0, \infty). \end{cases} \tag{4.1}$$

Suppose that x_1 and x_2 are solutions of (4.1) *for different initial conditions $\varphi = \varphi_1$ and $\varphi = \varphi_2$ respectively. Then,*

$$\lim_{t \to \infty} |x_1(t) - x_2(t)| = 0.$$

The proof will occupy the remainder of this section. We will first make some reductions. Let x_1 and x_2 be solutions for initial conditions φ_1 and φ_2. Define $y = x_2 - x_1$. For $t \le 0$, $y(t) = \varphi_2(t) - \varphi_1(t) = \psi(t)$, and ψ is bounded and continuous. For $t \ge 0$, we may subtract the delay differential equations satisfied by x_1 and x_2 to obtain

$$x_2'(t) - x_1'(t) + \int_0^\infty [g(t, s, x_2(t), x_2(t-s))) - g(t, s, x_1(t), x_1(t-s))]\, d\mu(s) = 0\,.$$

We may rewrite this as

$$y'(t) + \int [g(t,s,y(t)+x_1(t),y(t-s)+x_1(t-s)) - g(t,s,x_1(t),x_1(t-s))]\,d\mu(s) = 0. \tag{4.2}$$

Thus, if we define h by

$$h(t,s,\xi,\eta) = g(t,s,\xi+x_1(t),\eta+x_1(t-s)) - g(t,s,x_1(t),x_1(t-s)),$$

we see that y is a solution of the initial value problem

$$\begin{cases} y(t) = \psi(t), & t \in (-\infty,0] \\ y'(t) + \displaystyle\int_0^\infty h(t,s,y(t),y(t-s))\,d\mu(s) = 0, & t \in [0,\infty). \end{cases}$$

By the $\mathcal{G}$-lemma, h is again in $\mathcal{G}$. We also have $h(t,s,0,0) \equiv 0$. Thus, in order to prove Theorem 4.1, it will suffice to prove the following proposition.

Proposition 4.2. *Suppose that $g \in \mathcal{G}$ and that $g(\cdot,\cdot,0,0) = 0$. Let μ be a finite positive Borel measure on $[0,\infty)$. Suppose that $\varphi \in BC(-\infty,0]$, and let x be the solution of the initial value problem*

$$\begin{cases} x(t) = \varphi(t), & t \in (-\infty,0] \\ x'(t) + \displaystyle\int_0^\infty g(t,s,x(t),x(t-s))\,d\mu(s) = 0, & t \in [0,\infty). \end{cases} \tag{4.3}$$

Then $\lim_{t\to\infty} x(t) = 0$.

In order to prove Proposition 4.2, we will reduce to the case $\varphi = 0$, as in the last section.

Thus, let $\bar\varphi$ be an extension of φ to $(-\infty,\infty)$. We choose $\bar\varphi$ so that it is continuously differentiable on $[0,\infty)$ and such that the support of $\bar\varphi$ is bounded above. Thus, both $\bar\varphi$ and $\bar\varphi'$ are bounded.

Let x be a solution of the initial value problem (4.3) and define $y(t) = x(t) - \bar\varphi(t)$. Then y is bounded and continuous, $y = 0$ on $(-\infty,0]$ and by substituting $x = y + \bar\varphi$ into (4.3), we see that for $t \geq 0$, y satisfies

$$y'(t) + \bar\varphi'(t) + \int_0^\infty g(t,s,y(t)+\bar\varphi(t),y(t-s)+\bar\varphi(t-s))\,d\mu(s) = 0.$$

We choose to rewrite this as

$$\begin{aligned} y'(t) + \int_0^\infty [g(t,s,y(t)+\bar\varphi(t),y(t-s)+\bar\varphi(t-s)) - g(t,s,\bar\varphi(t),\bar\varphi(t-s))]\,d\mu(s) \\ = -\bar\varphi'(t) - \int_0^\infty g(t,s,\bar\varphi(t),\bar\varphi(t-s))\,d\mu(s). \end{aligned} \tag{4.4}$$

We claim that the right hand side of this equation goes to 0 as t goes to infinity. This is certainly true of $\bar\varphi'$, since the support of $\bar\varphi$ is bounded above.

To deal with the other term, hold s fixed for a moment. Then, for sufficiently large t, both $\bar\varphi(t)$ and $\bar\varphi(t-s)$ are zero. Since $g(t,s,0,0) \equiv 0$, we conclude that

$g(t, s, \bar{\varphi}(t), \bar{\varphi}(t - s)) = 0$ for all sufficiently large t. Thus, in the integral

$$\int_0^\infty g(t, s, \bar{\varphi}(t), \bar{\varphi}(t - s))\, d\mu(s) \tag{4.5}$$

the integrand goes to zero for each fixed s as $t \to \infty$. The integrand is also bounded in absolute value by some constant, and constant functions are integrable with respect to μ, since μ is finite. Thus, (4.5) goes to zero as $t \to \infty$ by the Dominated Convergence Theorem.

If we now define h by

$$h(t, s, \xi, \eta) = g(t, s, \xi + \bar{\varphi}(t), \eta + \bar{\varphi}(t - s)) - g(t, s, \bar{\varphi}(t), \bar{\varphi}(t - s)), \tag{4.6}$$

h is again in $\mathcal{G}$ and $h(t, s, 0, 0) \equiv 0$. Thus, y is the solution of an initial value problem

$$\begin{cases} y(t) = 0, & t \in (-\infty, 0] \\ y'(t) + \displaystyle\int_0^\infty h(t, s, y(t), y(t - s)) = f(t), & t \in [0, \infty), \end{cases} \tag{4.7}$$

where $f(t) \to 0$ as $t \to \infty$.

If we show that the solution of y of (4.7) goes to zero at infinity, then the solution $x = y + \bar{\varphi}$ of (4.3) will also go to zero at infinity, since the support of $\bar{\varphi}$ is bounded above.

Thus, in order to prove Proposition 4.2, it will suffice to prove the following proposition.

Proposition 4.3. *Suppose that $g \in \mathcal{G}$ and $g(\cdot, \cdot, 0, 0) = 0$. Let μ be a finite positive Borel measure on $[0, \infty)$. Suppose that $f \in BC[0, \infty)$ and $f(t) \to 0$ as $t \to \infty$. Let x be the solution of the initial value problem*

$$\begin{cases} x(t) = 0, & t \in (-\infty, 0] \\ x'(t) + \displaystyle\int_0^\infty g(t, s, x(t), x(t - s))\, d\mu(s) = f(t), & t \in [0, \infty). \end{cases} \tag{4.8}$$

Then $\lim\limits_{t \to \infty} x(t) = 0$.

The proof of Proposition 4.3 will occupy the rest of the section.

Using the apparatus of the last section, our strategy is as follows. Let $X_0 \subseteq X$ be the space of functions $x \in X$ such that $x(t) \to 0$ as $t \to \infty$. It is easy to check that X_0 is a closed subspace of X.

Suppose that we show that X_0 is invariant under all of the operators T_a. Then, as in the last section, we may find some ρ such that $B(\rho)$ is invariant under $T_{a(\rho)}$ and $T_{a(\rho)}$ is a contraction on $B(\rho)$. Thus, $T_{a(\rho)}$ has a fixed point x in $B(\rho)$, which is precisely the solution of the initial value problem (4.8). But then $X_0 \cap B(\rho)$ is a closed subset of X which is invariant under $T_{a(\rho)}$ and on which $T_{a(\rho)}$ is a contraction. Thus, the fixed point x must be in X_0, i.e., $x(t) \to 0$ as $t \to \infty$.

Let $BC_0[0, \infty)$ denote the space of functions $f \in BC[0, \infty)$ such that $f(t) \to 0$ as $t \to \infty$. Since $T_a(x) = L_a^{-1} M_a(x) + L_a^{-1} f$, to show that X_0 is invariant under T_a, it will suffice to show that M_a sends X_0 into $BC_0[0, \infty)$ and L_a^{-1} sends $BC_0[0, \infty)$ into X_0 (since f in (4.8) is in $BC_0[0, \infty)$).

To show that $M_a(x)$ sends X_0 into $BC_0[0,\infty)$, it will plainly suffice to show that the operator H defined by

$$H(x)(t) = \int_0^\infty g(t,s,x(t),x(t-s))\,d\mu(s), \qquad t \geq 0\,, \tag{4.9}$$

sends X_0 into $BC_0[0,\infty)$. To show this, suppose that $x \in X_0$ and choose $\rho \geq \|x\|$. Since $g(\cdot,\cdot,0,0) = 0$, (2.3) gives

$$|g(t,s,x(t),x(t-s))| \leq K(0,\rho)\max\{\,|x(t)|,|x(t-s)|\,\}\ .$$

If we hold s fixed and let t go to infinity, $x(t)$ and $x(t-s)$ go to zero, and so $g(t,s,x(t),x(t-s)) \to 0$. Thus, the integrand in (4.9) goes to zero as $t \to \infty$ for fixed s. Since the integrand is bounded by a constant, the Dominated Convergence Theorem shows that $H(x)(t) \to 0$ as t goes to infinity.

To show that L_a^{-1} maps $BC_0[0,\infty)$ into X_0, suppose that $f \in BC_0[0,\infty)$. By our formula for L_a^{-1}, we have

$$L_a^{-1}f(t) = \int_0^t e^{a(s-t)}f(s)\,ds\,.$$

for $t \geq 0$. By a simple change of variable, we may rewrite this convolution as

$$L_a^{-1}f(t) = \int_0^t e^{-as}f(t-s)\,ds\,.$$

Thus, we have

$$|L_a^{-1}f(t)| \leq \int_0^t e^{-as}|f(t-s)|\,ds \leq \int_0^\infty e^{-as}|f(t-s)|\,ds\,,$$

where the last integral converges because f is bounded. In the integral

$$\int_0^\infty e^{-as}|f(t-s)|\,ds\,, \tag{4.10}$$

the integrand goes to zero as $t \to \infty$ for each fixed s, and the integrand is bounded by the integrable function $e^{-as}\|f\|$. Thus, (4.10) goes to zero as t goes to infinity, and so $L_a^{-1}f$ goes to zero at infinity.

This completes the proof of Proposition 4.3 and hence the proof of Theorem 4.1.

5. Exponential Asymptotic Stability

In this section, we show that under an exponential decay condition on the the measure μ, the solutions of our initial value problem are globally exponentially asymptotically stable. Specifically, we prove the following theorem.

Theorem 5.1. *Suppose that $g \in \mathcal{G}$ and that μ is a finite Borel measure on $[0,\infty)$ such that*

$$\int_0^\infty e^{\lambda_0 s}\,d\mu(s) < \infty, \qquad \textit{for some } \lambda_0 > 0. \tag{5.1}$$

For $f \in BC[0,\infty)$ and $\varphi \in BC(-\infty,0]$, consider the initial value problem

$$\begin{cases} x(t) = \varphi(t), & t \in (-\infty,0] \\ x'(t) + \displaystyle\int_0^\infty g(t,s,x(t),x(t-s))\,d\mu(s) = f(t), & t \in [0,\infty). \end{cases} \tag{5.2}$$

Then, if x_1 and x_2 are two solutions of (5.2) *for different initial conditions $\varphi = \varphi_1$ and $\varphi = \varphi_2$ respectively, then there are constants $C \geq 0$ and $\lambda > 0$ such that*

$$|x_1(t) - x_2(t)| \leq Ce^{-\lambda t}, \qquad t \geq 0.$$

To prove this theorem, we let $y = x_2 - x_1$ and make the same reduction we made in Section 4 in going from Theorem 4.1 to Proposition 4.2. Thus, to prove Theorem 5.1, it will suffice to prove the following proposition.

Proposition 5.2. *Suppose that $g \in \mathcal{G}$ and that μ is a finite Borel measure on $[0, \infty)$ which satisfies the condition* (5.1)*. Suppose, also, that $g(\cdot, \cdot, 0, 0) = 0$. If $\varphi \in BC(-\infty, 0]$ and x is the solution of the initial value problem*

$$\begin{cases} x(t) = \varphi(t), & t \in (-\infty, 0] \\ x'(t) + \displaystyle\int_0^\infty g(t, s, x(t), x(t-s))\, d\mu(s) = 0, & t \in [0, \infty), \end{cases} \tag{5.3}$$

then there are constants $C \geq 0$ and $\lambda > 0$ such that

$$|x(t)| \leq Ce^{-\lambda t}, \qquad t \geq 0.$$

To prove this proposition, we next make the same reduction that we made in Section 4 in passing from Proposition 4.2 to Proposition 4.3. Thus, suppose that x is the solution of (5.3). Choose an extension $\bar{\varphi}$ of φ which is C^1 on $[0, \infty)$ and has support bounded above. Define $y = x - \bar{\varphi}$, so $y = 0$ on $(-\infty, 0]$. As before, we may write the equation satisfied by y in the form (4.4). In our present context, we need to show that the function on the right hand side of (4.4) is exponentially decreasing. This is is no problem for $\bar{\varphi}'$, since the support of $\bar{\varphi}$ is bounded above. Thus, we need the following lemma.

Lemma 5.3. *Suppose that $g \in \mathcal{G}$ and that $g(\cdot, \cdot, 0, 0) = 0$. Let μ satisfy Condition* (5.1)*. Define a function f by*

$$f(t) = \int_0^\infty g(t, s, \bar{\varphi}(t), \bar{\varphi}(t-s))\, d\mu(s)\,, \tag{5.4}$$

where $\bar{\varphi} \in BC(-\infty, \infty)$ and the support of $\bar{\varphi}$ is bounded above by $b > 0$. Then there is a constant $C \geq 0$ such that

$$|f(t)| \leq Ce^{-\lambda_0 t}, \qquad t \geq 0\,.$$

Proof of Lemma. Consider first the μ-measure of the interval $[t, \infty)$. Let K denote the value of the integral in (5.1). Then, for any $t \geq 0$ we have

$$\begin{aligned} e^{\lambda_0 t} \mu[t, \infty) &= \int_t^\infty e^{\lambda_0 t}\, d\mu(s) \\ &\leq \int_t^\infty e^{\lambda_0 s}\, d\mu(s) \\ &\leq \int_0^\infty e^{\lambda_0 s}\, d\mu(s) = K, \end{aligned}$$

and so

$$\mu[t, \infty) \leq Ke^{-\lambda_0 t}.$$

Next consider the function $g(t, s, \bar\varphi(t), \bar\varphi(t - s))$. If $t > b$, this function will be zero if $t - s > b$, since $g(t, s, 0, 0) \equiv 0$. Thus, the integrand in (5.4) is nonzero only for $s \geq t - b$. Then we have

$$|f(t)| \leq \int_{t-b}^{\infty} |g(t, s, \bar\varphi(t), \bar\varphi(t - s))| \, d\mu(s).$$

The integrand is bounded by some constant C, so we have

$$|f(t)| \leq C\mu[t - b, \infty) \leq CKe^{\lambda_0 b} e^{-\lambda_0 t},$$

for $t > b$. Since $|f|$ is bounded on $[0, b]$, this completes the proof of the lemma. □

To return to the discussion prior to the lemma, by applying this lemma, we see that $y = x - \bar\varphi$ is a solution of an initial value problem of the form (4.7), where h is defined by (4.6) and f is exponentially decreasing. If we show that the solution of this initial value problem is exponentially decreasing, $x = y + \bar\varphi$ will be exponentially decreasing, since the support of $\bar\varphi$ is bounded above. Thus, to prove Proposition 5.2, it will suffice to prove the following proposition.

Proposition 5.4. *Suppose that $g \in \mathcal{G}$ and $g(\cdot, \cdot, 0, 0) = 0$. Let μ be a finite positive Borel measure on $[0, \infty)$ that satisfies (5.1). Suppose that $f \in BC[0, \infty)$ and that there is a constant K such that $|f(t)| \leq Ke^{-\lambda_0 t}$. Let x be the solution of the initial value problem*

$$\begin{cases} x(t) = 0, & t \in (-\infty, 0] \\ x'(t) + \displaystyle\int_0^{\infty} g(t, s, x(t), x(t - s)) \, d\mu(s) = f(t), & t \in [0, \infty). \end{cases} \tag{5.5}$$

Then, for $\lambda > 0$ sufficiently small, there is a constant C such that

$$|x(t)| \leq Ce^{-\lambda t}.$$

To prove this proposition, we first introduce some spaces of exponentially decreasing functions. For $\lambda > 0$, let $Z(\lambda)$ denote the space of functions $x \in X$ such that $|x(t)| \leq Ce^{-\lambda t}$ for some constant C and all t (recall that elements of X are zero on $(-\infty, 0]$).

We will be able to show that $Z(\lambda)$ is invariant under the operators T_a for sufficiently small λ. This is not sufficient to prove the proposition, since $Z(\lambda)$ is not closed in X. Indeed, if $X_c \subseteq X$ denotes the set of functions in X with compact support, then $X_c \subseteq Z(\lambda) \subseteq X_0$ for all λ, but the closure of X_c in X is X_0. To deal with this problem, we put a norm on $Z(\lambda)$ and show that an appropriate T_a is a contraction in this norm.

Let $e_\lambda(t) = e^{\lambda t}$. A function $x \in X$ is in $Z(\lambda)$ if and only if $e_\lambda x$ is bounded. We define the norm on $Z(\lambda)$ by

$$\|x\|_\lambda = \|e_\lambda x\| = \sup \left\{ e^{\lambda t} |x(t)| \mid t \in \mathbb{R} \right\}.$$

Since $e_\lambda \geq 1$ where $x \neq 0$, we see that $\|x\| \leq \|x\|_\lambda$, so the inclusion of $Z(\lambda)$ into X is continuous. The mapping $Z(\lambda) \to X \colon x \mapsto e_\lambda x$ is a (bijective) isometry, so $Z(\lambda)$ is a Banach space. If $\lambda_1 < \lambda_2$, then $e^{\lambda_1 t} \leq e^{\lambda_2 t}$ where x is not zero, so $\|x\|_{\lambda_1} \leq \|x\|_{\lambda_2}$. Thus, $Z(\lambda_2) \subseteq Z(\lambda_1)$ and the inclusion is continuous.

By the definition of the norm, if $x \in Z(\lambda)$, we have

$$|x(t)| \leq \|x\|_\lambda e^{-\lambda t}.$$

We make similar definitions for spaces of exponentially decreasing functions on $[0,\infty)$. Thus, $Z_+(\lambda)$ will denote that space of functions $f \in BC[0,\infty)$ such that $e_\lambda f$ is bounded and we equip $Z_+(\lambda)$ with the norm $\|f\|_\lambda = \|e_\lambda f\|$.

We next make some estimates for our operators on the spaces of exponentially decreasing functions.

We first consider the operators L_a^{-1}. Suppose that $f \in Z_+(\lambda)$, where $\lambda < a$. Then $L_a^{-1}f(t)$ is zero for $t \le 0$, and for $t \ge 0$, we have

$$\begin{aligned} |L_a^{-1}f(t)| &\le \int_0^t e^{-as}|f(t-s)|\,ds \\ &\le \int_0^t e^{-as}\|f\|_\lambda e^{-\lambda(t-s)}\,ds \\ &= \|f\|_\lambda e^{-\lambda t}\int_0^\infty e^{-as}e^{\lambda s}\,ds. \end{aligned}$$

The last integral has the value

$$\frac{1}{a-\lambda}[1-e^{-(a-\lambda)t}],$$

which is less than $1/(a-\lambda)$, since $a-\lambda > 0$. Thus, we have

$$\|L_a^{-1}f\|_\lambda \le \frac{1}{a-\lambda}\|f\|_\lambda, \qquad f \in Z_+(\lambda),\ \lambda < a. \tag{5.6}$$

We next turn to the operators N_a. If $x, y \in Z(\lambda) \cap B(\rho)$ and $(t,s) \in [0,\infty) \times [0,\infty)$, we may apply Lemma 2.2 to conclude

$$\begin{aligned} &|[ax(t) - g(t,s,x(t),x(t-s))] - [ay(t) - g(t,s,y(t),y(t-s))]| \\ &\le \quad K(a,\rho)\max\{\,|x(t)-y(t)|, |x(t-s)-y(t-s)|\,\} \\ &\le \quad K(a,\rho)\max\left\{\,\|x-y\|_\lambda e^{-\lambda t}, \|x-y\|_\lambda e^{-\lambda(t-s)}\,\right\} \\ &= \quad K(a,\rho)\|x-y\|_\lambda e^{-\lambda t}e^{\lambda s}. \end{aligned}$$

Thus, for $x, y \in Z(\lambda) \cap B(\rho)$,

$$|N_a(x)(t,s) - N_a(y)(t,s)| \le K(a,\rho)\|x-y\|_\lambda e^{-\lambda t}e^{\lambda s}\,. \tag{5.7}$$

Now consider the operator $M_a\colon X \to BC[0,\infty)$. Suppose that $x, y \in Z(\lambda) \cap B(\rho)$, where $\lambda \le \lambda_0$. Then we have

$$\begin{aligned} |M_a(x)(t) - M_a(y)(t)| &\le \int_0^\infty |N_{a/m}(x)(t,s) - N_{a/m}(y)(t,s)|\,d\mu(s) \\ &\le \int_0^\infty K(a/m,\rho)\|x-y\|_\lambda e^{-\lambda t}e^{\lambda s}\,d\mu(s) \\ &= K(a/m,\rho)\|x-y\|_\lambda e^{-\lambda t}\int_0^\infty e^{\lambda s}\,d\mu(s)\,, \end{aligned}$$

where the last integral is finite because $\lambda \le \lambda_0$. In particular, if we set $y = 0$ and note that $N_a(0) = 0$ and $M_a(0) = 0$ (because $g(\cdot,\cdot,0,0) = 0$), we see that $M_a(x) \in Z_+(\lambda)$ if $x \in Z(\lambda)$ and $\lambda \le \lambda_0$. We also conclude that for $x, y \in Z(\lambda) \cap B(\rho)$ and $\lambda \le \lambda_0$,

$$\|M_a(x) - M_a(y)\|_\lambda \le K(a/m,\rho)\|x-y\|_\lambda \int_0^\infty e^{\lambda s}\,d\mu(s)\,. \tag{5.8}$$

Consider the initial value problem (5.5), so $f \in Z_+(\lambda_0)$. Assume that $\lambda \leq \lambda_0$ and $\lambda < a$. If $x \in Z(\lambda)$ then, from the results above, $M_a(x) \in Z_+(\lambda)$, $f \in Z_+(\lambda)$ and $L_a^{-1} M_a(x)$ and $L_a^{-1} f$ are in $Z(\lambda)$. Thus, $Z(\lambda)$ is invariant under the operator $T_a(x) = L_a^{-1} M_a(x) + L_a^{-1} f$. For $x, y \in Z(\lambda) \cap B(\rho)$, we have the estimate

$$\|T_a(x) - T_a(y)\|_\lambda \leq \frac{1}{a-\lambda} K(a/m, \rho) \|x - y\|_\lambda \int_0^\infty e^{\lambda s}\, d\mu(s)\,. \tag{5.9}$$

We know that we can find a $\rho > 0$ such that $T_{a(\rho)}(B(\rho)) \subseteq B(\rho)$. If we fix such a ρ and assume $\lambda < a(\rho)$ and $\lambda < \lambda_0$, the set $Z(\lambda) \cap B(\rho)$ is invariant under $T_{a(\rho)}$. The set $Z(\lambda) \cap B(\rho)$ is closed in $Z(\lambda)$, because the inclusion of $Z(\lambda)$ into X is continuous. For $x, y \in Z(\lambda) \cap B(\rho)$, we have

$$\|T_{a(\rho)}(x) - T_{a(\rho)}(y)\| \leq \frac{1}{a(\rho) - \lambda} K(a(\rho)/m, \rho) \|x - y\|_\lambda \int_0^\infty e^{\lambda s}\, d\mu(s).$$

Thus, $T_{a(\rho)}$ is Lipschitz on $Z(\lambda) \cap B(\rho)$ with Lipschitz constant

$$\sigma(\lambda) = \frac{1}{a(\rho) - \lambda} K(a(\rho)/m, \rho) \int_0^\infty e^{\lambda s}\, d\mu(s).$$

But σ is a continuous, nondecreasing, function of λ and

$$\sigma(0) = \frac{1}{a(\rho)} K(a(\rho)/m, \rho) m,$$

which we know is strictly less than one. Thus, $\sigma(\lambda) < 1$ for $\lambda > 0$ sufficiently small.

We conclude that if we choose λ sufficiently small that $\lambda < \lambda_0$, $\lambda < a(\rho)$ and $\sigma(\lambda) < 1$, then $T_{a(\rho)}$ leaves $Z(\lambda) \cap B(\rho)$ invariant and is a contraction on this closed subset of $Z(\lambda)$. Thus, $T_{a(\rho)}$ has a fixed point $x \in Z(\lambda)$, which is precisely the solution of the initial value problem (5.5). This completes the proof of Proposition 5.4 and hence the proof of Theorem 5.1.

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