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PERIODIC SOLUTIONS OF POLYNOMIAL NON-AUTONOMOUS DIFFERENTIAL EQUATIONS

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Abstract. We present some results on the number of periodic solutions for scalar non-autonomous polynomial equations of degree five. We also consider a class of polynomial equations of any degree. Our results give upper bounds for the number of limit cycles of two-dimensional systems.

1. Introduction

We consider differential equation

$$\dot{z} := \frac{dz}{dt} = P_0(t)z^n + P_1(t)z^{n-1} + \cdots + P_{n-1}(t)z + P_n(t) \tag{1.1}$$

where z is a complex-valued function and P_i are real-valued continuous functions. This class of equations has received some attention in the literature. The number of periodic solutions of such equations has been studied in [3, 5, 6, 8, 9, 10, 11, 12, 13].

We denote by $z(t, c)$ the solution of (1.1) satisfying $z(0, c) = c$. Take a fixed real number ω, we define the set Q to be the set of all complex numbers c such that $z(t, c)$ is defined for all t in the interval $[0, \omega]$; the set Q is an open set. On Q we define the displacement function q by

$$q(c) = z(\omega, c) - c.$$

Zeros of q identify initial points of solutions of (1.1) which satisfy the boundary conditions $z(0) = z(\omega)$. We describe such solutions as *periodic* even when the functions P_i are not themselves periodic. However, if P_i are ω-periodic then these solutions are also ω-periodic.

Note that q is holomorphic on Q. The multiplicity of a periodic solution φ is that of $\varphi(0)$ as a zero of q. It is useful to work with a complex dependent variable. The reason is that periodic solutions cannot then be destroyed by small perturbations of the right-hand side of the equation. Suppose that φ is a periodic solution of multiplicity k. By applying Rouche's theorem to the function q, for any sufficiently small perturbations of the equation, there are precisely k periodic solutions in a neighborhood of φ (counting multiplicity). Upper bounds to the number of periodic solutions of (1.1) can be used as upper bounds to the number of periodic solutions

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when z is limited to be real-valued. This is the reason that P_i are not allowed to be complex-valued.

When $n = 3$, equation (1.1) is known as the Abel differential equation. This case is of particular interest because of a connection with Hilbert's sixteenth problem; see [5] for details. It was shown in [9] and [12] that when $P_0(t) = 1$, then Abel differential equation has exactly three periodic solutions provided account is taken of multiplicity. However, local questions related to Hilbert's sixteenth problem (bifurcation of small-amplitude limit cycles and center conditions) are reduced to polynomial equations in which P_0 does have zeros. In this case the results of [9] and [12] no longer hold; indeed Lins Neto [8] has given examples which demonstrate that there is no upper bound for the number of periodic solutions. These examples can be used to show that there is no upper bound to the number of periodic solutions when $n \geq 4$ and $P_0(t) = 1$. On the other hand, systems with constant angular velocities can be reduced to polynomial equations. Global results about the number of periodic solutions can be used to obtain information about the number of limit cycles; see, for example, [4] and [7]

The case $n = 4$ was considered in [3] and [6] with $P_0(t) = 1$. The main concern was the multiplicity of $z = 0$ when the coefficients are polynomial functions in t and in $\cos t$ and $\sin t$. It was shown in [6] that the multiplicity is at most 8 when the coefficients are of degree 2; this result provides a counterexample to Shahshahani conjecture [13]. In [3], the methods of Groebner bases were used to study multiplicity and bifurcation of periodic solutions. In particular, it was shown that the multiplicity is at most 10 when the coefficients are polynomial functions of degree 3.

In this paper, we consider the case $n = 5$. The aim is to gain information on the total number of periodic solutions; this is a global question, while looking at multiplicity leads only to local results. In Section 2, we describe the phase portrait of (1.1) and recall some results from [9]. In Section 3, we present some results on the number of periodic solutions. In Section 4, we consider the real equation and from the derivatives of the displacement function, we deduce some results on the number of real periodic solutions. We also consider a class of equations with $n \geq 5$ and give an upper bound to the number of real periodic solutions. This result generalizes a recent result of Panov [11]. In the final Section, we return to polynomial two-dimensional systems. We use the results of Sections 3 and 4 to give upper bounds for the number of limit cycles.

2. The Phase Portrait

If $\varphi(t)$ is a periodic solution of

$$\dot{z} = z^5 + P_1(t)z^4 + P_2(t)z^3 + P_3(t)z^2 + P_4(t)z + P_5(t) \tag{2.1}$$

we make the transformation $z \mapsto z - \varphi(t)$; (2.1) then becomes

$$\dot{z} = z^5 + P_1(t)z^4 + P_2(t)z^3 + P_3(t)z^2 + P_4(t)z. \tag{2.2}$$

If φ is real, the coefficients of (2.2) are also real. No generality is lost by considering equation (2.2) because (2.1) has at least one real periodic solution. In fact, if n is odd and $P_0 \equiv 1$ then equation (1.1) has at least one real periodic solution. This result was given in [12] for equations with periodic coefficients. It can be verified quite easily that the method of proof in [12] works whether the coefficients are periodic or not.

We identify equation (2.2) with the quadruple (P_1, P_2, P_3, P_4) and write $\mathcal{L}$ for the set of all equations of this form. With the usual definitions of additions and scalar multiplications, $\mathcal{L}$ is a linear space; it is a normed space if for $P = (P_1, P_2, P_3, P_4)$, we define

$$\|P\| = \max\{\max_{0\le t\le\omega} |P_1(t)|, \max_{0\le t\le\omega} |P_2(t)|, \max_{0\le t\le\omega} |P_3(t)|, \max_{0\le t\le\omega} |P_4(t)|\}$$

The displacement function q is holomorphic on the open set Q. Since $z = 0$ is a solution, Q contains the origin. Moreover, q depends continuously on P with the above norm on $\mathcal{L}$ and the topology of uniform convergence on compact sets on the set of holomorphic functions.

The positive real axis and the negative real axis are invariant. Moreover, if φ is a non-real solution which is periodic, then so is $\bar{\varphi}$, its complex conjugate.

In [9], it was shown that the phase portrait of (2.2) is as shown in Figure 1 below. We refer to [9] for the details. There, the coefficients $P_i(t)$ were $\omega-$periodic. It can be verified that the same methods are applicable to the study of the number solutions that satisfy $z(0) = z(\omega)$ whether the coefficients are periodic or not.

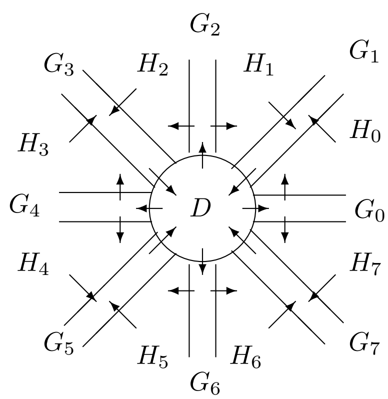

FIGURE 1. Phase Portrait

Note that the radius, ρ, of the disc D depends only on $\|P\|$ and ω. If $z = re^{i\theta}$ then the sets $G_k, k = 0, 1, \dots, 7$, which are the arms in the figure, are defined by

$$G_k = \{z | r > \rho, \frac{k\pi}{4} - \frac{a}{r} < \theta < \frac{k\pi}{4} + \frac{a}{r}\}$$

where $a = \max\{6, 6\|P\|\}$. Between the arms are the sets $H_k, k = 0, 1, \dots, 7$, which are defined by

$$H_k = \{z | r > \rho, \frac{k\pi}{4} + \frac{a}{r} \le \theta \le \frac{(k+1)\pi}{4} - \frac{a}{r}\}$$

For even k, trajectories can enter G_k only across $r = \rho$, and for odd k, trajectories can leave G_k only across $r = \rho$. No solution can become infinite in H_k as time either increases or decreases. Every solution enters D. Solutions become unbounded if and only if they remain in one of the arms G_k, tending to infinity as t increases if k is even and as t decreases if k is odd. For each k, there is a unique curve C_k on the bottom of G_k such that the solution $z(t, c)$ remains in G_k for as long as it is defined if and only if $c \in C_k$.

Let $q(P,c) = z_P(\omega,c) - c$, where $z_P(t,c)$ is the solution of $P \in \mathcal{L}$ satisfying $z_P(0,c) = c$. Suppose that (P_j) and (c_j) are sequences in $\mathcal{L}$ and $\mathbb{C}$, respectively, such that $q(P_j, c_j) = 0$. If $P_j \to P$ and $c_j \to c$ as $j \to \infty$, then either $q(P,c) = 0$, in this case $z_P(t,c)$ is a periodic solution, or $z_P(t,c)$ is not defined for the whole interval $0 \le t \le \omega$. In the later case, we say that $z_P(t,c)$ is a *singular periodic solution*. We also say that P has a singular periodic solution if $c_j \to \infty$; in this case there are τ and c such that the solution z_P with $z_P(\tau) = c$ becomes unbounded at finite time as t increases and as t decreases. We summarize the results of [9] when applied to (2.2).

Proposition 2.1. (i) *Let $\mathcal{A}$ be the subset of $\mathcal{L}$ consisting of all equations which have no singular periodic solutions. The set $\mathcal{A}$ is open in $\mathcal{L}$. All equations in the same components of $\mathcal{A}$ have the same number of periodic solutions.*
(ii) *The equation $\dot{z} = z^5$ has exactly five periodic solutions.*
(iii) *The number of periodic solutions of equation* (2.2) *is odd.*
(iv) *For each k, there is just one solution that crosses $r = \rho$ at a given time and becomes infinite without leaving G_k (under reversed time if k is odd).*

3. Number of Periodic Solutions

We call the solution $z = 0$ a *center* if $z(t,c)$ is periodic for all c in a neighborhood of 0. When P_0 has zeros then there are equations with a center. For cubic equations, this is related to the classical center problem of polynomial two-dimensional systems; we refer to [5] for details. However, when P_0 has no zeros then $z = 0$ is never a center. This result was proven in [6] for the case $n = 4$. We give a brief proof, with $n = 5$, for the sake of completeness.

Theorem 3.1. *The solution $z = 0$ is isolated as a periodic solution of* (2.2).

Proof. Suppose, if possible, that there is a open set $U \subset \mathbb{C}$ containing the origin such that all solutions starting in U are periodic. Then $q \equiv 0$ in the component of its domain of definition containing the origin. But the real zeros of q are contained in the disc D. Thus

$$\inf\{c \in \mathbb{R} : c > 0, z(t,c) \text{ is not defined for } 0 \le t \le \omega\} < \infty$$

It follows that there is a real singular periodic solution; but a positive real periodic solution which tends to infinity can do so only as t increases. This is a contradiction, and the result follows. □

Now, we give the result about the number of periodic solutions.

Theorem 3.2. *Suppose that $r^3 - rP_2(t) - P_3(t) \ge 0$ and $r^3 - rP_2(t) + P_3(t) \ge 0$ for positive r and $0 \le t \le \omega$. Then equation* (2.2) *has exactly five periodic solutions.*

Proof. With $z = re^{i\theta}$, we have

$$\dot{\theta} = r^4 \sin 4\theta + r^3 P_1(t) \sin 3\theta + r^2 P_2(t) \sin 2\theta + r P_3(t) \sin\theta.$$

If $|c| > \rho$ and is real then the real solution $z(t,c)$ remains outside the disk D as t increases and will become infinite. Solutions that enter G_0 or G_4 will leave G_0 or G_4, except the solution that enters at the intersection of C_0 and C_4 with the real axis; this solution is real because any solution which is once real is always real. Therefore, the unique solution that becomes infinite described in part (iv) of Proposition 2.1 is a real solution if $k = 0$ or $k = 4$. On the other hand, no real solution is unbounded as t increases and decreases. Hence, no singular periodic solution enters G_0 or G_4 because singular periodic solutions are unbounded both as t increases and decreases. Thus, a singular periodic solution enters D from G_1 or G_3 and leaves D to G_2. Hence, for a singular periodic solution $\dot{\theta} > 0$ at $\theta = \frac{\pi}{3}$ and $\dot{\theta} < 0$ at $\theta = \frac{2\pi}{3}$. On the other hand,

$$\begin{gathered}
\dot{\theta}(\frac{\pi}{3}) = \frac{-\sqrt{3}}{2} r(r^3 - rP_2(t) - P_3(t)), \\
\dot{\theta}(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2} r(r^3 - rP_2(t) + P_3(t))
\end{gathered}$$

Under the above hypotheses, $\dot{\theta}(\frac{\pi}{3}) < 0$ and $\dot{\theta}(\frac{2\pi}{3}) > 0$. Therefore, no singular periodic solution can enter D from G_1 or G_3 and leaves D to G_2. Since the phase portrait is symmetric about the $x-$axis, no singular periodic solution can enter D from G_5 or G_7 and leaves D to G_6. It follows that the equation does not have a singular periodic solution.

Now, consider the class of equations

$$\dot{z} = z^5 + sP_1(t)z^4 + sP_2(t)z^3 + sP_3(t)z^2 + sP_4(t)z,$$

with $0 \le s \le 1$. If $r^3 - rP_2(t) - P_3(t) \ge 0$ and $r^3 - rP_2(t) + P_3(t) \ge 0$ then $r^3 - srP_2(t) - sP_3(t) \ge 0$ and $r^3 - srP_2(t) + sP_3(t) \ge 0$ for $0 \le s \le 1$. Therefore, any equation in this family does not have singular periodic solutions. The equation $\dot{z} = z^5$ belongs to this family and has five periodic solutions. By part (i) of Proposition 2.1, each of these equations has five periodic solutions. □

Corollary 3.3. *If $P_3(t) \equiv 0$, and $P_2(t) \le 0$ then* (2.2) *has five periodic solutions.*

4. Real Periodic Solutions

Consider the equation

$$\dot{x} = f(x,t)$$

where $x \in \mathbb{R}$ and f is as smooth as is required in the argument. With $f_k = \frac{\partial^k f}{\partial x^k}$, we define

$$\begin{gathered}
E(t,c) = \exp\big[\int_0^t f_1(x(t,c),\tau)d\tau\big], \\
D(t,c) = E(t,c) f_2(x(t,c),t), \\
G(t,c) = \int_0^t D(\tau,c)d\tau
\end{gathered}$$

From [8], we have the following formulae for the first three derivatives of $q(c)$,

$$\begin{gathered}
q'(c) = E(\omega,c) - 1, \\
q''(c) = E(\omega,c)\int_0^\omega D(t,c)dt, \\
q'''(c) = E(\omega,c)[\frac{3}{2}(G(\omega,c))^2 + \int_0^\omega (E(\omega,c))^2 f_3(x(t,c),t)dt]\,.
\end{gathered}$$

Formulae for the fourth and fifth derivatives of q are given in [6]. Their use is not as direct as that of the first three derivatives; simply $f_4 \ge 0$ does not imply that $q^{(iv)} \ge 0$.

If the n-th derivative of a function does not change sign on an interval, then the function has at most n zeros in that interval. Using this fact and the formulae for the derivatives of q, we prove the following.

Theorem 4.1. *Consider the equation*

$$\dot{x} = x^5 + P_1(t)x^4 + P_2(t)x^3 + P_3(t)x^2 + P_4(t)x + P_5(t) \tag{4.1}$$

with $x \in \mathbb{R}$.
(i) If $P_2(t) \geq 0.4(P_1(t))^2$ *then* (4.1) *has at most three real periodic solutions.*
(ii) If $P_1(t) \geq 0$, $P_2(t) \geq 0$, *and* $P_3(t) \geq 0$, *then* (4.1) *has at most two positive periodic solutions.*

Proof. (i) Since $f_3 = 6(10x^2+4P_1x+P_2)$, it follows that $f_3 \geq 0$ if $16(P_1)^2-40P_2 \leq 0$. Hence, $q'''(c) > 0$ if $P_2 \geq 0.4P_1^2$. Therefore, q has at most three zeros.

(ii) The conditions imply that $f_2 \geq 0$ for positive x. This implies that $q''(c) > 0$ when $c > 0$. □

Now, we consider the equation

$$\dot{x} = x^n + P_1(t)x^m + P_2(t)x^3 + P_3(t)x^2 + P_4(t)x + P_5(t), \tag{4.2}$$

with $n > m > 3$. Using the ideas of cross-ratio, it was shown in [11] that equation (4.2) has at most three periodic solutions when n is odd and $P_1 \equiv P_2 \equiv 0$. The method used in the proof of Theorem 4.1. can be used to prove the following generalization of this result.

Theorem 4.2.
- (i) *If n and m are odd, $P_1(t) \geq 0$, and $P_2(t) \geq 0$ then* (4.2) *has at most three periodic solutions.*
- (ii) *If $P_1(t) \geq 0$, $P_2(t) \geq 0$, and $P_3(t) \geq 0$ then equation* (4.2) *has at most two positive periodic solutions.*
- (iii) *If n and m are odd, $P_1 \geq 0$, $P_2(t) \geq 0$ and $P_3(t) \leq 0$ then* (4.2) *has at most two negative periodic solutions.*
- (iv) *If $P_1(t) \geq 0$ and $P_2(t) \geq 0$ then* (4.2) *has at most three positive periodic solutions.*
- (v) *If n and m are even, $P_1 \geq 0$, $P_2(t) \equiv 0$, and $P_3(t) \geq 0$ then* (4.2) *has at most two real periodic solutions.*

5. Number of Limit Cycles

Consider the system

$$\begin{aligned} \dot{x} &= \lambda x - y + x(R_{n-1}(x,y) + R_{n-2}(x,y) + \dots + R_1(x,y)) \\ \dot{y} &= x + \lambda y + y(R_{n-1}(x,y) + R_{n-2}(x,y) + \dots + R_1(x,y)), \end{aligned} \tag{5.1}$$

where R_i is a homogeneous polynomial of degree i. The system in polar coordinates becomes

$$\dot{r} = r^n R_{n-1}(\cos\theta,\sin\theta) + r^{n-1}R_{n-2}(\cos\theta,\sin\theta) + \dots + r^2R_1(\cos\theta,\sin\theta) + \lambda r$$
$$\dot{\theta} = 1.$$

Some necessary conditions for a center are given in [4]. It is clear that the origin is the only critical point and if it is a center then it is a uniformly isochronous center. Limit cycles of (5.1) correspond to positive $2\pi-$periodic solutions of

$$\frac{dr}{d\theta} = R_{n-1}r^n + R_{n-2}r^{n-1} + \dots + R_1r^2 + \lambda r$$

Now, we consider the case $n = 5$. In the special case $R_3 \equiv 0$ and $R_1 \equiv 0$, the center conditions were given in [1, 2, 14]. The following result follows from Theorems 3.2 and 4.1.

Theorem 5.1. *Consider system* (5.1) *with $n = 5$ and $R_4 \equiv 1$.*

(i) *If $c^3 - cR_2(\cos\theta, \sin\theta) - R_1(\cos\theta, \sin\theta) \geq 0$, and $c^3 - cR_2(\cos\theta, \sin\theta) + R_1(\cos\theta, \sin\theta) \geq 0$ for positive c and $0 \leq \theta \leq 2\pi$, then the system has at most four limit cycles.*

(ii) *If $R_2(\cos\theta, \sin\theta) \geq 0.4(R_3(\cos\theta, \sin\theta))^2$, then the system has at most two limit cycles.*

(iii) *If $R_1 \equiv R_3 \equiv 0$, and $R_2(\cos\theta, \sin\theta) \geq 0$, then the system has at most two limit cycles.*

Finally, we consider the case

$$
\begin{aligned}
\dot{x} &= \lambda x - y + x(R_{n-1}(x, y) + R_{m-1}(x, y) + R_2(x, y) + R_1(x, y)) \\
\dot{y} &= x + \lambda y + y(R_{n-1}(x, y) + R_{m-1}(x, y) + R_2(x, y) + R_1(x, y)),
\end{aligned}
\tag{5.2}
$$

with $n > m > 3$. In polar coordinates, this system reduces to

$$
\frac{dr}{d\theta} = R_{n-1}r^n + R_{m-1}r^m + R_2 r^3 + R_1 r^2 + \lambda r.
$$

If a function R_i does not change sign, then it is necessary to assume that i is even. The following result follows directly from Theorem 4.2.

Theorem 5.2. (i) *Suppose that n and m are odd numbers, and $R_{n-1} \equiv 1$. If $R_{m-1}(\cos\theta, \sin\theta) \geq 0$ and $R_2(\cos\theta, \sin\theta) \geq 0$ then system* (5.2) *has at most two limit cycles.*

(ii) *If $R_{n-1} \equiv 1$, $R_{m-1}(\cos\theta, \sin\theta) \geq 0$, and $R_2(\cos\theta, \sin\theta) \geq 0$, then system* (5.2) *has at most three limit cycles.*

Remark 5.3. If the leading coefficient R_{n-1} does not vanish anywhere then the transformation of the independent variable

$$
\theta \mapsto \exp\Big(\int_0^\theta R_{n-1}(\cos u, \sin u)du\Big)
$$

reduces the polar equation into a similar equation but with a leading coefficient equals one.

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